Finite-Dimensional Frame Theory over Arbitrary Fields

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Background

Definition

A frame is a family of vectors $\mathcal{F} = \{f_1, \ldots, f_k\}$ in a Hilbert space $\mathcal{H}$ such that there exists $0 < A \leq B < \infty$ such that

$$A||x||^2 \leq \sum_{i=1}^{k} |\langle x, f_i \rangle|^2 \leq B||x||^2.$$

If $A = B = 1$, we say it is a Parseval frame.

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Reconstruction Formula: For a frame $\mathcal{F}$, there exists a set of vectors $\{g_i\}_{i=1}^{k}$ s.t. for all x in $\mathcal{H}$,

$$x = \sum_{i=1}^{k} \langle x, g_i \rangle f_i = \sum_{i=1}^{k} \langle x, f_i \rangle g_i.$$

We say $\{\mathrm{f}_i\}$ and $\{g_i\}$ are dual frames for $\mathcal{H}$.

Vector spaces over $\mathbb{Z}_2$

Dot product ceases to be a definite inner product in $\mathbb{Z}_2^n$

Example: $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 1 + 1 = 2 \equiv 0 \pmod 2$.

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Motivation: Establish a theory for frames without relying on definite inner products

Previous Work:

“Frame theory for binary vector spaces"- Bodmann et. al. (2009)

“Binary Frames" - Hotovy/Scholze/Larson (2010)

Indefinite Inner Product Spaces

Definition

$(V, \langle\cdot,\cdot\rangle)$ is an (indefinite) inner product space if $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{F}$ is a bilinear form (or sesquilinear if $\mathbb{F} = \mathbb{C}$).

Example:

The dot product is a bilinear map $\langle\cdot,\cdot\rangle : \mathbb{Z}_2^n \times \mathbb{Z}_2^n \to \mathbb{Z}_2$ given via

$$\left\langle \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \right\rangle = \sum_{i=1}^{n} a_i b_i.$$

Definition (Bodmann, et al. (2009))

A frame in a vector space V over a field $\mathbb{F}$ is a spanning set of vectors for V.

Riesz Representation Theorem

Theorem (Hotovy/Scholze/Larson 2011)

Let V, K be vector spaces over $\mathbb{Z}_2$ with a dual frame pair $\{x_i\}_1^k$, $\{y_i\}_1^k$. Then if $\phi : V \to K$ is a linear functional, then there exists a unique $z \in V$ such that $\phi(x) = \langle x, z \rangle$ for all $x \in V$.

Corollary (Existence of Adjoint)

There exists $\phi^* : K \to V$ such that $\langle \phi(x), y \rangle = \langle x, \phi^*(y) \rangle$ for all $x \in V$, $y \in K$. If $\phi = \phi^*$, we say ϕ is a self-adjoint operator.

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Note, not all subspaces of $\mathbb{Z}_2^n$ have dual frames:

Let $V = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$. Note that the dot product of any two vectors in V is zero, so there is no Riesz Representation theorem.

Analysis Operator

Definition (Hilbert space)

The analysis operator for a frame $\{f_i\}_{i=1}^k$ in a Hilbert space $\mathcal{H}$ is the map $\Theta : \mathcal{H} \to \mathbb{C}^k$ defined by $\Theta(x) = \left(\langle x, f_1\rangle, \ldots, \langle x, f_k\rangle\right)^T$.

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In a general vector space setting, what is the connection between the analysis operator and frames?

Definition

Let V be a finite-dimensional vector space over $\mathbb{F}$. We say the linear functionals $\{\phi_1, \ldots, \phi_k\}$ separate V if $\Theta(x) = (\phi_1(x), \ldots, \phi_k(x))^T$ is injective.

A Reconstruction Formula

Theorem

Let V be a n-dimensional space over a field $\mathbb{F}$. Let $\{\phi_1, \ldots, \phi_k\}$ separate V, i.e. Θ is injective. Then there exists a set of vectors $\{X_1, \ldots, X_k\} \subset V$ such that for all $x \in V$ we have that

$$x = \sum_{i=1}^{k} \overline{\phi_i(x)} X_i.$$

Analysis Spaces

Definition

A frame $\{x_i\}_{i=1}^k$ is an analysis frame for a vector space V if $\Theta : V \to \mathbb{F}^k$ defined by

$$\Theta(x) = (\langle x, x_1\rangle, \langle x, x_2\rangle, \ldots, \langle x, x_k\rangle)^T$$

is injective where $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ is an indefinite inner product.

Definition

$(V, \langle \cdot, \cdot \rangle)$ is called an analysis space if it admits an analysis frame.

We want to classify all such analysis spaces $(V, \langle \cdot, \cdot \rangle)$ over a field $\mathbb{F}$

Results on Analysis Spaces

Theorem

Let $\{x_i\}_{i=1}^k$ *be an analysis frame for a n-dimensional vector space V. Let* $E = Ran(\Theta) \subseteq \mathbb{F}^k$*. Then there exists a dual frame* $\{y_i\}_{i=1}^k$ *such that for all* $x \in V$*,*

$$x = \sum_{i=1}^{k} \langle x, x_i \rangle y_i = \sum_{i=1}^{k} \langle x, y_i \rangle x_i$$

where

$$x_i = \Theta^*(e_i), \quad y_i = \Theta^{-1}|_E P_E(e_i)$$

where $\{e_i\}$ *is the standard orthonormal basis for* $\mathbb{F}^k$*,* $\Theta^{-1}|_E$ *is the invertible map from E back to V, and* $P|_E$ *is an idempotent projection (i.e. not necessarily self-adjoint) onto E.*

$E = Ran(\Theta)$ admits a Parseval frame

Suppose we have an analysis frame $\{x_i\}_{i=1}^k$ for V. Suppose in addition, there exists a $\{z_i\}_{i=1}^k \subset V$ such that $\{\Theta(z_i)\}_{i=1}^k$ is a Parseval frame for $E = Ran(\Theta)$, i.e. we have a reconstruction formula given for all $u \in E$ by:

$$u = \sum_{i=1}^{k} \langle u, \Theta(z_i) \rangle \Theta(z_i).$$

Then we have that

$$x_i = \Theta^*(e_i)$$

and

$$y_i = \sum_{j=1}^{k} \langle e_i, \Theta(z_j) \rangle z_j$$

where $e_i, i = 1, \ldots, k$ is the standard basis for $\mathbb{F}^k$.

ZIP(V) and Analysis Spaces

We introduce the following subspace of V:

Definition

The zero inner product subspace of V is given by:

$$ZIP(V) := \{x \in V | \langle x, y \rangle = 0, \ \forall y \in V\}.$$

Example: Let $V = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$. Then $ZIP(V) = V$.

We formulate a useful characterization of analysis spaces:

Lemma

$(V, \langle \cdot, \cdot \rangle)$ *is an analysis space if and only if* $ZIP(V) = \{0\}$.

Equivalent Properties of Analysis Spaces

Theorem

Let $(V, \langle \cdot, \cdot \rangle)$ be an analysis space. Then the following are equivalent:

1. *V has a Riesz Representation theorem*
2. *V has a dual basis pair*
3. *All frames in V are analysis frames*
4. *V has at least one analysis frame*
5. $ZIP(V) = \{0\}$

Corollary

If $(V, \langle \cdot, \cdot \rangle)$ is a definite inner product space, then it is an analysis space.

Vector Space Decomposition

Theorem

Let V be a finite-dimensional vector space over $\mathbb{F}$. Then V can be written as the algebraic direct sum of an analysis space E and the space $ZIP(V)$, i.e.

$$V = (E \oplus ZIP(V), \langle \cdot, \cdot \rangle) = (E, \langle \cdot, \cdot \rangle_E) \oplus (ZIP(V), \langle \cdot, \cdot \rangle_{ZIP(V)})$$

where

$$\langle (e_1, z_1), (e_2, z_2) \rangle = \langle e_1, e_2 \rangle_E + \langle z_1, z_2 \rangle_{ZIP(V)}$$

for $e_1, e_2 \in E, \;\; z_1, z_2 \in ZIP(V)$.

Corollary

$V/ZIP(V)$ *is unitarily equivalent to* E, *i.e. there exists an isomorphism* $U : V/ZIP(V) \to E$ *such that* $\langle w_1, w_2 \rangle = \langle Uw_1, Uw_2 \rangle$ *for all* $w_1, w_2 \in V/ZIP(V)$.

A Finer Vector Space Decomposition

Let $V = E \oplus ZIP(V)$ where E is an analysis space.

Definition

Let E be an analysis space as given above. Let

$$Z_0 := \{x \in E | \langle x, x \rangle = 0 \text{ and } \langle x, y \rangle + \langle y, x \rangle = 0, \ \forall y \in E\}.$$

Theorem

Let V finite-dimensional vector space over $\mathbb{F}$ where $\mathbb{F} \neq \mathbb{C}$. Then

$$V = E' \dot{+} Z_0 \dot{+} ZIP(V)$$

where Z_0 and $ZIP(V)$ are defined as before and E' is an analysis space.

Note that $\langle \cdot, \cdot \rangle_V$ restricted to the analysis space E' becomes a definite inner product on E'.

References

1. Bernhard G. Bodmann, My Le, Matthew Tobin, Letty Reza and Mark Tomforde, Frame theory for binary vector spaces, Involve 2 589-602 (2009)
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