VISUALIZING $\mathcal{A}$-DISCRIMINANT VARIETIES AND THEIR TROPICALIZATIONS

JOANN CORONADO, SAMUEL PÉREZ-AYALA, AND BITHIAH YUAN

Abstract. The $\mathcal{A}$-discriminant variety, $\nabla_{\mathcal{A}}$, is the irreducible algebraic hypersurface describing all polynomials with singular complex zero sets and exponent sets contained in $\mathcal{A}$. The connected components of the real complement of $\nabla_{\mathcal{A}}$ (under log-absolute value) are called *discriminant chambers*, and are regions in coefficient space where the topology of the real zero set is constant. The Horn-Kapranov Uniformization provided us an efficient parametrization of $\nabla_{\mathcal{A}}$. Moreover, the tropical $\mathcal{A}$-discriminant is a polyhedral approximation to the image of the log-absolute value applied to $\nabla_{\mathcal{A}}$, and is an important first step toward computationally tractable approximations of discriminant chambers. Such approximations are central in providing results on the topology of real zero sets and faster homotopies preserving the number of real roots. Understanding the real solutions of polynomial equations has applications in numerous disciplines such as robotics and game theory. We are developing two software packages, one for visualizing the $\mathcal{A}$-discriminant chambers and one for computing their tropicalizations when $\mathcal{A}$ has cardinality $n + 4$.

1. Introduction

The use of polynomial models can be found in a number of areas including robotics, mathematical biology, game theory, statistics and machine learning. Furthermore, many of the models describing the physical world involve solving systems of real polynomial equations. However, polynomial systems whose real roots lie outside the reach of current algorithmic techniques are commonly found in industry. Many of these problems involve sparse polynomials with few terms. Though the number of terms may be few, these polynomials can be of high dimension. Abel's Theorem states that, for polynomials of degree 5 or higher, it is not possible to express the general solutions of polynomial equations in terms of radicals. This theorem points to the need for more general iterative algorithms that go beyond taking radicals.

1.1. **Sturm Sequences.**

In the 19th century, Sturm sequences were used as a method to find information on the number of real roots of a polynomial between two points. To implement this algorithm for any $f \in \mathbb{R}[x_i]$ of degree d, we define the Sturm sequence to be $P_f := (p_0, \cdots, p_d)$, where $p_0 := f, p_1 := f', p_i := q_{i+1}p_{i+1} - p_{i+2}$ for all $i \in \{0, \cdots, d-2\}$, and q_{i+1} and $-p_{i+2}$ are the quotient and remainder, respectively, obtained from dividing p_i by p_{i+1}. Also, define $P_f(c) := (p_0(c), \cdots, p_d(c))$ for any $c \in \mathbb{R}$ and $V_f(c)$ to be the number of sign changes in the sequence $P_f(c)$. Then, let $\sigma : \mathbb{R} \to \{-1, 0, 1\}$ be the sign function, which maps all positive numbers to 1, negative numbers to -1, and 0 to 0. The number of roots between points a and b can then be found by computing $V_f(a) - V_f(b)$.

Date: July 24, 2014.

Key words and phrases. $\mathcal{A}$-Discriminant, Horn Kapranov Uniformization, Amoeba.

Example 1.1. *To solve for the number of roots between* -3 *and* 3 *for the polynomial,* $f(x) = x^4 - 2x^2 + 1$*, we calculate the Sturm sequence,* $P_f(x) := (x^4 - 2x^2 + 1, 4x^3 - 4x, x^2 - 1, 0, 0)$*. Inputting* -3 *and* 3 *gives us*

$\sigma(P_f(-3)) = (1, -1, 1)$ *and* $\sigma(P_f(3)) = (1, 1, 1)$*. Because* $V_f(-3) - V_f(3) = 2$ *, we know there are 2 real roots between -3 and 3 for* f*.*

The algorithm offers a relatively simple method to calculate the real roots between two points. However, the polynomials in the Sturm sequence can contain coefficients with hundreds of thousands of digits when the given polynomial is in high dimensions. These results can occur even when the number of terms of the polynomial is relatively small. Therefore, we look to other methods for information of the real roots of a polynomial.

2. Preliminaries

It can be useful, when dealing with sparse polynomials, to classify polynomials based on the number of variables to the number of terms rather than on its degree. For example, the polynomial, $f(x, y) = c_0x^3 + c_1x^2y + c_2y^3 + c_3$, can be classified as a cubic polynomial based on the degree. $f(x, y)$ can also be classified as a bi-variate, 4-nomial. We can classify any polynomial as an n-variate, $(n + k)$-nomial, where n is the number of variables and $(n + k)$ is the number of terms. To study the real zero set of an n-variate $n + k$-nomial , we can first focus on finding the degenerate roots.

Definition 2.1. $\zeta \in \mathbb{C}$ is a degenerate root of $f(x_1, x_2, ..., x_n)$ if

$$f(\zeta) = \frac{\partial f}{\partial x_1}(\zeta) = \frac{\partial f}{\partial x_2}(\zeta) = \cdots = \frac{\partial f}{\partial x_n}(\zeta) = 0$$

We can find which polynomials within a family of polynomials have degenerate roots where the family is represented by a support.

Definition 2.2. Given $f(x_1, x_2, ..., x_n) = \sum_{i=1}^{t} c_i x^{a_i}$ where t is the number of terms, $c_i \in \mathbb{C}$, and $a_i \in \mathbb{Z}^n$. The support of f, is $\mathcal{A} = \{a_1, \ldots, a_t\}$.

Then, the polynomials with degenerate roots are represented by the $\mathcal{A}$-discriminant variety.

Definition 2.3. Fix $\mathcal{A} = \{a_1, a_2, ...a_{n+k}\} \subseteq \mathbb{Z}^n$. Then the $\mathcal{A}$-discriminant variety, denoted by $\nabla_{\mathcal{A}}$, is defined as the closure of

$$\{(c_1, c_2, ..., c_{n+k}) \in (\mathbb{C}^*)^{n+k} \mid f(x) = \sum_{i=1}^{n+k} c_i x^{a_i} \textit{ has a degenerate root}\}$$

Furthermore, the $\mathcal{A}$-discriminant polynomial, denoted by $\Delta_{\mathcal{A}} \in \mathbb{Z}[c_i, \cdots, c_{n+k}]$, is defined to be (up to sign) the irreducible defining polynomial of ∇_A.

Example 2.4. *Given* $c_0x^2 + c_1x + c_2$, $\mathcal{A} = \{2, 1, 0\}$,

$\Delta_{\mathcal{A}} = c_1^2 - 4c_0c_2$ *and each element in*

$\nabla_{\mathcal{A}}$ *is a solution to the equation* $c_1^2 - 4c_0c_2 = 0$ *These elements, such as* $(2, 4, 2)$ *and* $(1, 6, 9)$*, correspond to polynomials with support,* $\mathcal{A}$ *with degenerate roots.* $(2, 4, 2)$*and* $(1, 6, 9)$ *represent* $2x^2 + 4x + 2$ *and* $x^2 + 6x + 9$ *respectively.*

3. Amoeba and Viro Diagrams

We can visualize $\nabla_{\mathcal{A}}$ by taking log-absolute value of the zero set of $\Delta_{\mathcal{A}}$.

Definition 3.1. For any polynomial of the form $f(x) = \sum_{i=1}^{n} c_i x^{a_i}$ its amoeba is defined to be $\{Log|x| \mid x_i \in \mathbb{C}^*, f(x) = 0\}$.

Important information can be found by studying the real complement of the amoeba.

Theorem 3.2 (Archimedean Amoeba Theorem [3])**.** *The complement of Amoeba(f) in $\mathbb{R}^n$ is a finite disjoint union of open convex sets. Each unbounded open convex set is called an outer chamber.*

The outer chambers of the amoeba are regions in coefficient space where the topology of the real zero set is constant. In each chamber the topology of the zero sets of the polynomials are isotopic, meaning that the real zero sets of the polynomials within each chamber can be continuously deformed into one another. Additionally, there is a one to one correspondence between the outer chambers and the triangulation for the polynomials within the chamber. These triangulations can help us to find the Viro diagrams of the polynomials within the corresponding chamber.

The Viro diagram for a polynomial, f, obtained from the traingulation in a chamber is isotopic to the topology of the real zero set of f. To construct the Viro diagram for f, we must find the convex hull and the newton polytope of f.

Definition 3.3. For N points $p_1, ..., p_N$, the convex hull C is given by the expression $C \equiv \sum_{j=1}^{N} \lambda_j p_j : \lambda_j \geq 0$ for all j and $\sum_{j=1}^{N} \lambda_j = 1$. In other words, the convex hull of S is the smallest convex set containing S, denoted Conv(S). Additionally, given $f(x_1, ..., x_n) = \sum_{i=1}^{t} c_i x^{a_i}$ for $c_i \in \mathbb{C} \setminus \{0\}$, the newton polytope is Newt(f) = Conv(Supp(f)). Also we can define triangulation of a point set S to be a simplicial complex $\sum$ whose vertices lie in S. Define archnewt(f) by archnewt(f) = Conv($\{(a_i, -\log|c_i|)\}_{i=1}^{t} \subseteq \mathbb{R}^{n+1}$). Finally, let the lower hull of archnewt(f) be the union of lower faces of archnewt(f).

Once we've obtained the triangulation, we label the vertices of the newton polytope of f based on the sign of the polynomial coefficient corresponding to each vertex. Between alternating signs, we mark the midpoints and, within each triangulation, take the convex hull of the midpoints. The set of convex hulls forms the Viro diagram.

4. Calculating the $\mathcal{A}$-Discriminant Polynomial

Viro diagrams allow us to study the zero set of the $\mathcal{A}$-discriminant polynomial using the triangulations within each chamber of $\log|\nabla_{\mathcal{A}}|$. The $\mathcal{A}$-discriminant of the quadratic case mentioned previously is one case of trinomials of the form $f(x) = c_1 + c_2 x^d + c_3 x^D$ with $0 < d < D$ and $c_1, c_2, c_3 \neq 0$. We can find the $\mathcal{A}$-discriminant polynomial for trinomials of this form by first finding xf' and setting both equations equal to zero. We can express these equations in the following form.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & d & D \end{bmatrix} \begin{bmatrix} c_1 \\ c_2x^d \\ c_3x^D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ where } \hat{\mathcal{A}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & d & D \end{bmatrix}$$

We can also find a generator, B, for the right null-space of $\hat{\mathcal{A}}$ which gives us the equations: $b_1 + b_2 + b_3 = 0$ and $db_2 + Db_3 = 0$. We can then express the generator in terms of D and d. So

$$\begin{bmatrix} c_1 \\ c_2x^d \\ c_3x^D \end{bmatrix} = \alpha \begin{bmatrix} D-d \\ -D \\ d \end{bmatrix}$$

We can manipulate these equations to obtain the $\mathcal{A}$-discriminant polynomial for our trinomials.

$$\left(\frac{c_1}{D-d}\right)^{D-d} \left(\frac{-c_2}{D}\right)^{-D} \left(\frac{c_3}{d}\right)^{d} - 1 = 0$$

Now, we will introduce two ways to determine the $\mathcal{A}$-discriminant polynomial for more complicated cases through resultants and the Horn-Kapranov Uniformization.

4.1. **Resultant.** In order to explain the resultant method, we need to explain what the Sylvester Matrix is.

Definition 4.1. If we have two univariate polynomials $f(x) = a_1x + a_2x^2 + \cdots + a_nx^n$ and $g(x) = b_1x + b_2x^2 + \cdots + b_mx^m$, then the **Sylvester Matrix** of f and g is the following $(n+m) \times (n+m)$ square matrix:

$$Syl(f,g) = \begin{pmatrix} a_n & a_{n-1} & \cdots & a_1 & 0 & \cdots & 0 \\ 0 & a_n & a_{n-1} & \cdots & a_1 & \cdots & 0 \\ \vdots & & & & & & \\ 0 & \cdots & 0 & a_n & a_{n-1} & \cdots & a_1 \\ b_m & b_{m-1} & \cdots & b_1 & 0 & \cdots & 0 \\ 0 & b_m & b_{m-1} & \cdots & b_1 & \cdots & 0 \\ \vdots & & & & & & \\ 0 & \cdots & 0 & b_m & b_{m-1} & \cdots & b_1 \end{pmatrix}$$

The resultant of two polynomials is the determinant of the Sylvester matrix. More formally:

Definition 4.2. The Resultant of two univariate polynomials f and g (as defined above) is the determinant of the Sylvester Matrix of f and g. That is,

$$Res(f,g) = \det(Syl(f,g))$$

A theorem from Gelfand, Kapranov and Zelevinski [3] explains why we can use the resultant to obtain the $\mathcal{A}$-discriminant polynomial.

Theorem 4.3. *Let $f(x) = a_1x + a_2x^2 + \cdots + a_nx^n$ and $g(x) = b_1x + b_2x^2 + \cdots + b_mx^m$. Then $Res(f,g) = 0 \iff f(x) = g(x) = 0$ has a common complex root or $a_n = b_m = 0$*

The $\mathcal{A}$-discriminant polynomial is a polynomial that vanishes whenever our polynomial and its first derivative have a common root. If we apply the resultant to a univariate polynomial and its derivative, we find that $\Delta_{\mathcal{A}}$ is a factor of the resultant.

However, as the number of terms increase, we can easily obtain huge polynomials via the resultant method. If we want to visualize $\nabla_{\mathcal{A}}$, we have to find the roots of $\Delta_{\mathcal{A}}$, but this is precisely our problem: finding the roots of polynomials in higher dimensions. We will introduce another way to determine the $\mathcal{A}$-discriminant called the Horn Kapranov Uniformization.

4.2. **Horn-Kapranov Uniformization.** The Horn-Kapranov Uniformization gives us an explicit parametrization of $\nabla_{\mathcal{A}}$ without knowing the $\mathcal{A}$-Discriminant polynomial.

Theorem 4.4 (M. Kapranov [4]). *Let $\mathcal{A} = \{a_1, \cdots, a_{n+k}\} \in \mathbb{Z}^n$ be the support for $f \in \mathcal{F}_{\mathcal{A}}$ and define the following matrix:*

$$\hat{A} = \begin{pmatrix} 1 & \cdots & 1 \\ a_1 & \cdots & a_{n+k} \end{pmatrix}$$

The parametrization of $\nabla_{\mathcal{A}}$ is given by the closure of

$$\nabla_{\mathcal{A}} = \left\{[\beta_1\lambda_1 t^{a_1} : \cdots : \beta_{n+k}\lambda_{n+k}t^{a_{n+k}}] |\ \beta \in \mathbb{C}^{n+k},\ \hat{A}B = 0,\ t \in (\mathbb{C}^*)^n\right\}$$

where β_i are the rows of B.

Example 4.5. *Considering the cubic case, $f(x) = c_0 + c_1x + c_2x^2 + c_3x^3$, we can parametrize $\nabla_{\mathcal{A}}$ as follows:*

$$\left\{[\lambda_1 t^0 : \lambda_2 t^1 : (-3\lambda_1 - 2\lambda_2)t^2 : (2\lambda_1 + \lambda_2)t^3] \mid \lambda_1, \lambda_2 \in \mathbb{C}, t \in \mathbb{C}^*\right\}$$

We can reduce the dimension of our polynomial to study the real part of $\nabla_{\mathcal{A}}$. Assuming that c_0 and c_2 are nonzero, $\frac{1}{c_0} f\left((\frac{c_0}{c_2})^{1/2}x\right)$ gives us roots that only differ by some scalar. Then, we reduce the study of $f(x)$ to the study of $1 + \gamma x + x^2 + \beta x^3$. We obtain the next two dimensional parametrization for $\nabla_{\mathcal{A}}$, denoted as $\bar{\nabla}_{\mathcal{A}}$:

$$\bar{\nabla}_{\mathcal{A}} = \left\{\left(\frac{\lambda_1(2\lambda_1 + \lambda_2)^2}{(-3\lambda_1 - 2\lambda_2)^3}, \frac{\lambda_2(2\lambda_1 + \lambda_2)}{(-3\lambda_1 - 2\lambda_2)^2}\right) \mid \lambda_1, \lambda_2 \in \mathbb{C}\right\}$$

In order to visualize our parametrization on logarithmic paper, *Amoeba*$(\Delta_{\mathcal{A}})$, we can define the following function $\varphi : \mathbb{R}^{k-1} \to \mathbb{R}^{n+k}$ as $\varphi(\lambda) := \mathrm{Log}|\lambda B^T|$. Here, B is a matrix whose columns form a basis for the right nullspace of $\hat{\mathcal{A}}$. In order to explain one important property of this function, we need to define the Minkowski sum.

Definition 4.6. For any two subsets $U, V \subseteq \mathbb{R}^n$, we define the Minkowski Sum $U + V$ to be the set $\{u + v \mid u \in U, v \in V\}$.

There is a corollary [1] that tells us that $Log|\nabla_{\mathcal{A}}|$ is the minskowski sum of the image of $\varphi(\lambda)$ and the row space of $\bar{\mathcal{A}}$.

We can generalize the reduction process by applying B as right-multiplication. Then we will obtain the following:

$$Log|\bar{\nabla}_{\mathcal{A}}| = Log|(\nabla_{\mathcal{A}})^B| = Log|\nabla_{\mathcal{A}}|B = Log|\lambda B^T|B$$

The map that gives us the parametrization of the reduced $\mathcal{A}$-discriminant variety is $\bar{\varphi}$: $\mathbb{P}^{k-2}_{\mathbb{R}} \setminus \mathcal{H}_B \to \mathbb{R}^{k-1}$ defined as $\bar{\varphi}(\lambda) = \log|\lambda B^T|B$.

Following **Example 4.5**, $\bar{\varphi}$ gives us: $\lambda_1, \lambda_2 \in \mathbb{P}^1_{\mathbb{R}} \setminus \{[0:1],[1:0],[2:-3],[-1:2]\}$:

$$(\log|\lambda_1| + 2\log|2\lambda_1+\lambda_2| - 3\log|3\lambda_1+2\lambda_2|,\ \log|\lambda_2| + \log|2\lambda_1+\lambda_2| - 2\log|3\lambda_1+2\lambda_2|)$$

4.2.1. *Hyperplane Arrangements.* There are many points where $\bar{\varphi}(\lambda)$ blow up to negative infinity. We can represent all those points using the concept of hyperplanes. A hyperplane $H \in \mathbb{R}^n$ is any set of the form

$$H = \{x \in \mathbb{R}^n \mid a_1x_1 + \cdots + a_nx_n = c\}$$

for $a \in \mathbb{R}^n$ and real number c. An arrangement of hyperplanes is a finite set $\mathcal{H}$ in a projective space $\mathbb{P}$.

In our particular case, we can define the hyperplane arrangement as follows:

Definition 4.7. Following the notation of **Theorem 4.4**,

$$\mathcal{H}_B = \{[\lambda \in \mathbb{P}^{k-2}_{\mathbb{R}}] \mid \lambda\beta_i = 0 \; for\ some\ i \in \{1,\cdots,n+k\}\}$$

is called the hyperplane arrangement corresponding to B [6].

Note that $\log|\lambda B^T|$ is undefined at the λ's that belong to $\mathcal{H}_B$. Then it is clear that, as $\bar{\varphi}$ approaches $\lambda \in \mathcal{H}_B$, $\bar{\varphi}$ blows up to negative infinity times each of the rows of B. The hyperplane arrangement of a given support provides us useful information to easily parametrize the $\mathcal{A}$-discriminant variety.

5. From Projective Space to the Reduced $\mathcal{A}$-Discriminant Ameoba

In this section we will explain what we did in our project to visualize the $\mathcal{A}$-discriminant variety when $\mathcal{A}$ has cardinality $(n+4)$.

There already exists a Sage code from Korben Rusek [7] that plots the contour of the reduced $\mathcal{A}$-discriminant amoeba for any support of cardinality $n+3$ where the contour is the image of the real zero set of a polynomial under the $\text{Log}|\cdot|$ map. Rusek also created an animation to visualize the quartic case. We are still developing a sofware package, using Sage, that will plot the contour of the reduced $\mathcal{A}$-discriminant amoeba for any support $\mathcal{A}$ with cardinality $n+4$.

Definition 5.1. Let $\mathbb{K}$ be any field. We define the projective space $\mathbb{P}$ over the field $\mathbb{K}$ as follows:

$$\mathbb{P}^n_{\mathbb{K}} := \{[z_0 : \cdots : z_n] \mid z_i \in \mathbb{K} \; not\ all\ zero\}$$

with the identification

$$[z_0 : \cdots : z_n] = [z_0\lambda : \cdots : z_n\lambda] \; for\ all\ \lambda \in \mathbb{K}^*$$

.

Most of the time, we use the projective space over the field of real numbers. For example, points in $\mathbb{P}^1_{\mathbb{R}}$, represent lines through the origin in $\mathbb{R}^2$. Lines in $\mathbb{P}^2_{\mathbb{R}}$, represent planes through the origin in $\mathbb{R}^3$.

Going back to **Example 4.5**, the hyperplane arrangement corresponding to B is $\{[0:1],[1:0],[2:-3],[-1:2]\}$. Note that all of the points in $\mathbb{P}^1_{\mathbb{R}}$ correspond to a set of lines through the origin in $\mathbb{R}^2$ where $\bar{\varphi}(\lambda)$ blow up to infinity. We can represent the four projective points using $\mathbb{S}^1$.

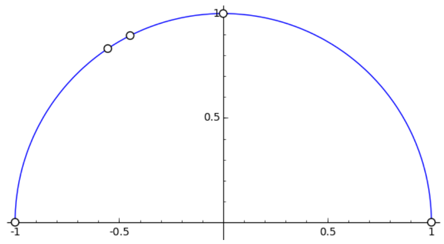

Figure 1: Projective points on $\mathbb{S}^1$ for **Example 4.5**

$\bar{\varphi}(\lambda)$ maps each arc segment of the unit semi-circle $\mathbb{P}^1_{\mathbb{R}}$ to a piece of the contour of the reduced $\nabla_{\mathcal{A}}$ in $\mathbb{R}^2$.

There is one important difference between the n-variate $(n+3)$-nomials and n-variate $(n+4)$-nomials. In the $n+4$-case, the basis for the right null-space of $\hat{\mathcal{A}}$ always consists of three columns. Instead of lines through the origin, $\bar{\varphi}(\lambda)$ has whole planes through the origin in $\mathbb{R}^3$ where our map blow up to the infinity. We can represent all those planes as lines in $\mathbb{P}^2_{\mathbb{R}}$.

We present an outline of the algorithm that we are working to implement in Sage to visualize the reduced $\mathcal{A}$-discriminant amoeba in $\mathbb{R}^3$.

Algorithm 5.2.

- *Input: Support $\mathcal{A} \subset \mathbb{Z}^n$ with $\#\mathcal{A} = n+4$*
- *Output: Surface of the reduced amoeba in $\mathbb{R}^3$*

(1) *Basis for the right null-space of $\hat{\mathcal{A}}$*
(2) *Determine the planes where $\bar{\varphi}(\lambda)$ blow up to infinity and represent it as circles (lines) in $\mathbb{P}^2$*
(3) *Find the intersection lines between the planes and represent it as points in $\mathbb{P}^2$*
(4) *Store all the information of vertices, edges and faces*

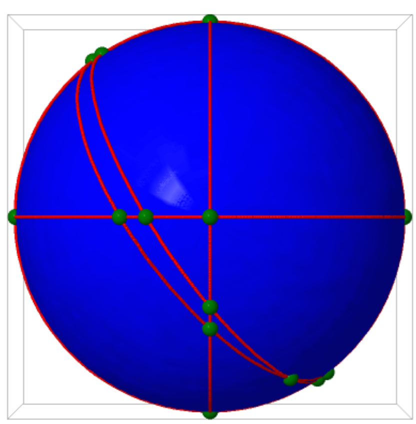

Figure 2: Projective lines and Points in $\mathbb{P}^2$ for the quartic case

After step (4) we can correspond any piece of the hemisphere to a piece of the reduced $\mathcal{A}$-discriminant amoeba. Then we can map all the pieces to the corresponding pieces of the reduced amoeba using $\bar{\varphi}(\lambda)$. **Figure 2** shows us the arrangement of lines, including intersection points, where $\bar{\varphi}(\lambda)$ blow up to negative infinity.

6. Computing $\mathcal{A}$-Discriminant Chambers

$\mathcal{A}$-discriminants are central in real root counting because the complement of the real part of $\nabla_{\mathcal{A}}$ determines where in coefficient space the real zero set of a polynomial changes topology. The connected components of the complement of the real part $Log|\nabla_{\mathcal{A}}|$ describe regions in coefficient space (called discriminant chambers) where the topology of the real zero set of a polynomial system is constant.

It follows that if we know which chamber our polynomial lives in, we can visualize the topology of the positive zero set of the polynomial and count the number of real roots. We will explore the methods that will help us compute the $\mathcal{A}$-discriminant chambers in the case where $\mathcal{A}$ has cardinality $(n+2), (n+3)$, and $(n+4)$. The main results of our project deal with computing the tropical $\mathcal{A}$-discriminant of $(n+4)$-nomials.

The formal definition of the discriminant chamber is as follows:

Definition 6.1. Suppose $\mathcal{A} = \{a_1, ..., a_{n+k}\} \subset \mathbb{Z}^n$ and $\nabla_{\mathcal{A}}$ is a hypersurface. Any connected component $\mathcal{C}$ of the complement of $\nabla_{\mathcal{A}}$ in $\mathbb{P}_{\mathbb{R}}^{n+k-1} \setminus \{c_1 \cdots c_{n+k} = 0\}$ is called a (real) discriminant chamber [1].

We define a (convex) cone in $\mathbb{R}^{n+k}$ to be any subset closed under nonnegative linear combinations. The cones will help us characterize subregions called the chamber cones where the number of real roots is easy to compute.

We start with finding the chambers for n-variate $(n+2)$-nomials.

6.1. Computing the $\mathcal{A}$-discrimiant chambers for n-variate (n+2)-nomials.

Let $f(x) = c_0 + c_1x + c_2x^2$. We reduce the $\mathcal{A}$-discriminant polynomial to $\frac{1}{4}$, which is just a point. It follows that we can parametrize quadratics with one number which is the point $\frac{1}{4}$ from the reduced $\mathcal{A}$-discriminant polynomial, $\overline{\Delta}_{\mathcal{A}}$. Then we evaluate $\Delta_{\mathcal{A}}$ at some given coefficients $c_1, ..., c_{n+2}$ to see if $\Delta_{\mathcal{A}}$ is positive, negative, or zero.

In any given orthant (selections of signs for the c_i) there are exactly 2 connected components for the complement of the zero set of $\Delta_{\mathcal{A}}$ and they are determined by the sign of $\Delta_{\mathcal{A}}$. If the polynomial is in the orthant where $c_0, c_1, c_2 > 0$ there are 2 chambers and each one correspond to a nonzero sign of $c_1^2 - 4c_0c_2$ or $c_0(\frac{c_1}{-2})^{-2}c_2 - 1$. Thus, c^B tells us that if the coefficient, c, of the polynomial is small then it lies on the left of $\frac{1}{4}$ and if c is big, then it lies on the right of $\frac{1}{4}$.

6.2. Computing $\mathcal{A}$-discriminant chambers for n-variate (n+3)-nomials.

Now, we will compute the chambers for n-variate $(n+3)$-nomials. We will see that we can approximate the amoeba of $\Delta_{\mathcal{A}}$ using the tropical $\mathcal{A}$-discriminant and the cutting-complex.

Definition 6.2. We call the facets of the (reduced) chamber cones of $\nabla_{\mathcal{A}}$ (reduced) walls of $\nabla_{\mathcal{A}}$. We also refer to walls of dimension 1 as rays [1].

Definition 6.3. When $\mathcal{A} \subset \mathbb{R}^n$ contains a point x such that $dimConv\mathcal{A} = 1 + dimConv(\mathcal{A} \setminus \{x\})$, we say that $Conv\mathcal{A}$ is a pyramid.We say that $\mathcal{A}$ is a near-circuit when $\mathcal{A}$ has cardinality $n+3$, $dimConv\mathcal{A} = n$, and $\mathcal{A}$ is not a pyramid [1].

Definition 6.4. Suppose $\mathcal{A} \subset \mathbb{Z}^n$ is a near-circuit. Also let B be any real $(n+3) \times 2$ matrix whose columns are a basis for the right null space of $\hat{\mathcal{A}}$, and let $\beta_1, ..., \beta_{n+3}$ be the rows of B. Any set of indices $\mathcal{J} \subset \{1, ..., n+3\}$ satisfying the two conditions:

(1) $[\beta_i]_{i \in \mathcal{J}}$ is a maximal rank 1 submatrix of B

(2) $\sum_{i \in \mathcal{J}} \beta_i$ is not the zero vector,

is called a radiant subset corresponding to $\mathcal{A}$ [1].

Theorem 6.5. *Suppose that $\mathcal{A} \subset \mathbb{Z}^n$ is a near-circuit. The number of chamber cones of $\nabla_{\mathcal{A}}$ and the number of radiant subsets corresponding to $\mathcal{A}$ are identical* [1].

By **Theorem 3.2**, each outer chamber of $\nabla_{\mathcal{A}}$ must be bounded by 2 walls, and the walls have a natural cyclic ordering. It follows that the number of chamber cones is the same as the number of rays. In particular, when $\mathcal{A} \subset \mathbb{Z}$ has cardinality $(n+3)$, $\nabla_{\mathcal{A}}$ is always a hypersurface.

We examined Algorithm 3.9 in [BHPR [1]] which inputs a near-circuit $\mathcal{A} \subset \mathbb{Z}^n$ of cardinality $n+3$ and the coefficient vector c of a given polynomial and outputs the chamber cone determined by the radiant subsets containing f. In particular, this algorithm checks the radiant subsets as they give us the rays that generate the chamber cones. The algorithm also computes the shifts of the rays by intersecting the lines to get a better approximation of $\nabla_{\mathcal{A}}$.

The union of the rays generated by the radiant subsets in this algorithm will give us the tropical $\mathcal{A}$-discriminant, $\tau(X^*_{\mathcal{A}})$, which approximates the chamber cones with the rays starting from the origin.

Definition 6.6. The tropical discriminant is the cone over the logarithmic limit set of $\Delta_{\mathcal{A}}$.

We can look at $\nabla_{\mathcal{A}}$ and find its amoeba by taking the $Log|\cdot|$. Then we can look at how the amoeba intersects a sphere. The intersections yield a union of pieces of the great hemispheres in the limit as the radius goes to infinity. Hence, if we connect the union of pieces to the origin we will get $\tau(X^*_{\mathcal{A}})$.

Shifting the rays will give us a better approximation of the amoeba which is referred to as the cutting-complex. Both the tropical $\mathcal{A}$-discriminant and the cutting-complex are polyhedral approximations of the $\mathcal{A}$-discriminant amoeba. Although the cutting-complex will give us a better approximation of $Amoeba(\overline{\Delta}_{\mathcal{A}})$ and its chambers, it is not always so easy to compute, especially when the number of monomial terms increases. Therefore, the tropical discriminant is an important step toward building the cutting-complex, which has nice approximation properties that are currently being investigated.

After identifying which chamber our polynomial lives in, our next step is to determine the topology of the positive zero set of the polynomial because this will help us count the number of real roots. We learned earlier that a triangulation of a point set $\mathcal{A}$ is a simplicial complex $\sum$ whose vertices lie in $\mathcal{A}$ (see **Definition 3.3**). From what we learned about the triangulations and Viro diagrams, we can visualize the topology of the positive zero set of a polynomial by plotting the archnewt(f) and look at the lower hull.

The GKZ-Correspondence tells us there is a 1-1 correspondence between the outer chambers and the topology of the real zero set, and the triangulations of the support $\mathcal{A}$. From the triangulations we can use Viro Diagrams to visualize the topology of the positive zero set of the polynomial. Thus, by computing which chamber cone contains our polynomial, we can easily visualize the topology of the positive zero set of the polynomials that lie in the chamber.

6.3. **Computing the Tropical $\mathcal{A}$-discriminant for n-variate $(n+4)$-nomials.**
Although we can extend parts of the algorithm for n-variate $(n+3)$-nomials to the n-variate $(n+4)$ case, computing which chamber the polynomial lives in is more involved. To help us with the process, we created an algorithm to compute the tropical $\mathcal{A}$-discriminant which is an important first step toward computationally tractable approximations of discriminant chambers. To give a more precise definition, the tropical $\mathcal{A}$-discriminant is a polyhedral approximation to the image of the log-absolute value applied to $\nabla_{\mathcal{A}}$. As we have seen in the $(n+3)$-case, such approximations starts at the origin and can help us easily identify the chambers cones. This will help us locate the chamber the polynomial lives in and is central in providing results on the topology of real zero sets and faster homotopies preserving the number of real roots via the GKZ-Correspondence.

We give an overview of the function of the tropical $\mathcal{A}$-discriminant, $\mathrm{Trop}(\Delta_{\mathcal{A}})$. $\mathrm{Trop}(\Delta_{\mathcal{A}}) \in \mathbb{R}^{n+k}$ approximates the amoeba of the $\mathcal{A}$-discriminant $\in \mathbb{R}^{n+k}$. After reducing $\Delta_{\mathcal{A}}$, $\mathrm{Trop}(\overline{\Delta}_{\mathcal{A}}) \in \mathbb{R}^{k-1}$ approximates the amoeba of the reduced $\mathcal{A}$-discrimiant $\in \mathbb{R}^{k-1}$. For example, in the case of 1-variate $(n+4)$-nomials, $\mathrm{Trop}(\overline{\Delta}_{\mathcal{A}}) \in \mathbb{R}^3$ approximates the amoeba of the reduced $\mathcal{A}$-discriminant $\in \mathbb{R}^3$.

The algorithm that we created computes the tropical $\mathcal{A}$-discriminant for the quartic.

Algorithm 6.7.

- *Input: $\mathcal{A} \subset \mathbb{Z}^n$ of cardinality $n+4$*
- *Output: Tropical $\mathcal{A}$-discriminant, $\tau(X^*_{\mathcal{A}})$*

(1) *Find the basis for the right null space B corresponding to $\hat{\mathcal{A}}$*
(2) *Compute the negative rows of B, denoted as $-\beta_i$*
(3) *Compute the intersections of the $-\beta_i$'s to find the vertices in $\mathcal{H}_B$*
(4) *Take the linear combination of the $-\beta_i$'s to find the cones*
(5) *Compute the 2-dimension cones that make up the walls corresponding to vertices of $\mathcal{H}_B$*
(6) *The tropical $\mathcal{A}$-discriminant is the union of the walls*

We saw earlier that in the case of $(n+4)$-nomials, the rows of B are vectors that make up the hyperplanes in the hyperplane arrangement from **Definition 4.7**. It follows that, the hyperplane arrangement in $\mathbb{R}^3$ gives us the locus of points where the logarithms blow up in $\mathbb{P}^2$. The Horn-Kapranov Uniformization (HKU) tells us that when λ approaches the line corresponding to β_i, HKU blows up in the direction of $-\beta_i$, which are the rays.

Definition 6.8. Let $\sum(\mathcal{H}_B)$ denote the corresponding polyhedral complex that partitions all of $\mathbb{P}^{k-2}_{\mathcal{C}}$. Let any $(k-2)$-dimensional cell $\sigma \in \sum(\mathcal{H}_B)$, and any vertex $v \in \sum(\mathcal{H}_B)$. Let W_v denote the cone generated by all $-\beta_i$ and β_i is a normal to a hyperplane of $\mathcal{H}_B$ incident to v. We call W_v a wall of $\mathcal{A}$.

It follows that the walls are the cones generated by the linear combination of the rays. There are several vertices, and each one determines a wall. We need to find the intersections of the rays to get the vertices, v, in $\mathcal{H}_B$ because the set of lines going through each vertex determines which rays make up the corresponding wall. In the case of a 1-variate $(n+4)$-nomial, each wall is a 2-dimensional cone. The walls are important because they will help us define the tropical discriminant.

Lemma 6.9. *The tropical discriminant, $\tau(X^*_{\mathcal{A}})$, is exactly the union of W_v over all vertices v of $\mathcal{H}_B$.*

It follows that $\tau(X_{\mathcal{A}}^*)$ is constructed by the union of W_v over all vertices v of $\mathcal{H}_B$.

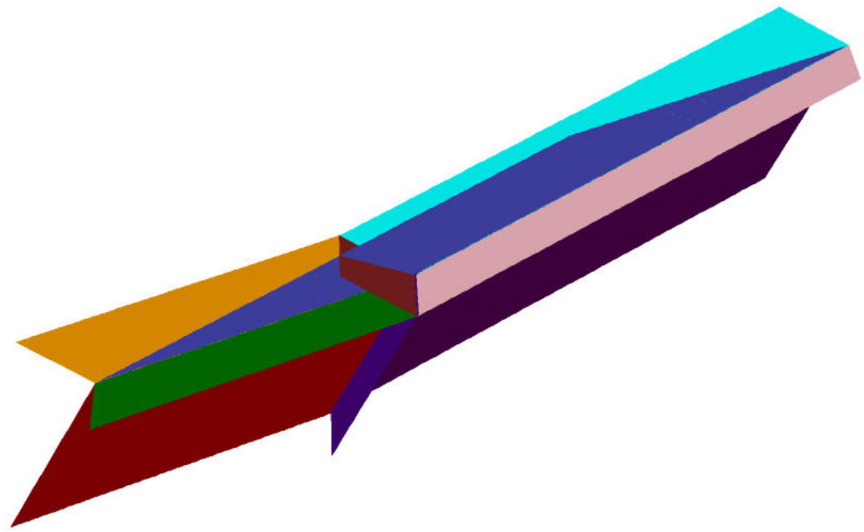

Figure 3: Tropical $\mathcal{A}$-Discriminant for the quartic

7. Future Work

We developed a software package in Sage that computes the tropical $\mathcal{A}$-discriminant for the quartic case. Next, we will be working on computing the tropical $\mathcal{A}$-discriminant for the general $(n+4)$ case. Then we want to be able to compute the reduced $\mathcal{A}$-discriminant amoeba, which is the image of cells of the spherical arrangement under the Horn-Kapranov Uniformization and compute the shifted walls. Ultimately, we want to develop a software package to quickly compute which $\mathcal{A}$-discriminant chamber contains the (n+4)-nomials. When given $\mathcal{A} \subset \mathbb{Z}^n$ of cardinality $n+4$ and the coefficient vector c of a given polynomial we want to identify which chamber cone contains f. We will also use triangulations and Viro Diagrams as tools to help us determine the topology which is constant in each discriminant chamber.

Appendix A.
Sage Code for **Algorithm 5.2**

```
#We input the support A as a list. For example, the
#support for the quartic case is written A = [[0,1,2,3,4]]
#The inputs and outputs of each function are listed below.
#Functions:
#1) get_B_list - input a support with cardinality n+4 and return the basis for
#the right null-space of A-hat.
#2) plane_arrangement - input a support with A=n+4, return the plot of the
#hyperplane arrangement corresponding to the B matrix with the unit sphere in R^3.
#3) sphere_lines - input a support with A = n+4, return the representation
#of the undefined planes only as lines on the unit sphere.
#4) get_parametric_eq - input a support with A=n+4, return the parametric
#equations of the circles that we can plot with sphere_lines function.
#5) intersection_points - input a support with A=n+4, return a plot of all the
#intersection points of any two circles on the upper half of the unit hemisphere.
#(the point represent the intersection lines between any two undefined plane in R^3).
#6) get_int_xyz - input a support with #A=n+4, return a list of all the intersection
#points in (x,y,z)-coordinates that represents the intersection lines between
#any undefined planes in R^3.
#7) get_int_lines - input a support with A=n+4, return a list of all pairs of lines
#with intersection.
##########################
def get_B_list(A):
   Ah = A + [[1]*len(A[0])]
   Am = matrix(Ah).transpose()
   Bm = Am.integer_kernel().basis_matrix().transpose()
   B = map(lambda b: list(b), list(Bm))

return B
##########################
def plane_arrangement(A):
    x,y,z=var('x,y,z')
    B = get_B_list(A)
    G=Graphics()
    G+=implicit_plot3d(x^2+y^2+z^2 == 1,(x,-1,1),(y,-1,1),(z,-1,1),color='blue')

    for i in range(len(B)):
             temp = B[i]
             G += implicit_plot3d(temp[0]*x +temp[1]*y +temp[2]*z==0,
                  (x,-1.2,1.2),(y,-1.2,1.2),(z,-1.2,1.2),color='red',thickness=20)
    return G
##########################
def sphere_lines(A):
    B=get_B_list(A)
    (u,v) = var('u,v')
```

```
    p1 = parametric_plot3d([cos(u)*cos(v),cos(v)*sin(u),sin(v)],(u,0,2*pi),
        (v, 0, 2*pi),plot_points=[50,50], aspect_ratio=[1,1,1],color = 'blue')
    p2 = parametric_plot3d([cos(u)*cos(v),cos(v)*sin(u),sin(v)],(u,0,2*pi),
        (v, 0, 2*pi),plot_points=[50,50], aspect_ratio=[1,1,1],color = 'blue')
    length=len(B)
    for i in range(length):
        n=B[i]
        f(x,y,z) = var('x,y,z')
        p1 = p1 + implicit_plot3d(n[0]*x + n[1]*y + n[2]*z==0, (x, -1, 1),
       (y, -1, 1), (z, 0, 1),aspect_ratio=1,color = 'red')
        N = vector([n[0],n[1],n[2]])
        L = N.column()
        C = L.integer_kernel().basis_matrix()
        U = vector(C[0])
        V = vector([N[1]*U[2] - N[2]*U[1], N[2]*U[0] - N[0]*U[2],
            N[0]*U[1] - N[1]*U[0]])
        U = U/norm(U)
        V = V/norm(V)
        t=var('t')
        p2 = p2 + parametric_plot3d([cos(t)*U[0]+sin(t)*V[0],cos(t)*U[1]+sin(t)*V[1],
             cos(t)*U[2]+sin(t)*V[2]],(t,0,2*pi),color = 'red',thickness=6)

    return p2
##########################
def get_parametric_eq(A):
    B = get_B_list(A)
    P=[]
    for i in range(len(B)):
        n=B[i]
        f(x,y,z) = var('x,y,z')
        N = vector([n[0],n[1],n[2]])
        L = N.column()
        C = L.integer_kernel().basis_matrix()
        U = vector(C[0])
        V = vector([N[1]*U[2] - N[2]*U[1], N[2]*U[0] - N[0]*U[2],
            N[0]*U[1] - N[1]*U[0]])
        U = U/norm(U)
        V = V/norm(V)
        t=var('t')
        P.append([cos(t)*U[0]+sin(t)*V[0],cos(t)*U[1]+sin(t)*V[1],
                  cos(t)*U[2]+sin(t)*V[2]])
    return P
##########################
def intersection_points(A):
    B = get_B_list(A)
    G = Graphics()
```

```
    length = len(B)
    for i in range(length):
        for j in range(length):
            if j > i:
                a = vector(B[i])
                b = vector(B[j])
                c = a.cross_product(b)
                if c.norm() !=0:
                     c = c / norm(c)
                     G+=point([c[0],c[1],c[2]],size=30,color='green')
                     G+=point([-1*c[0],-1*c[1],-1*c[2]],size=30,color='green')
    return G
##########################
def get_int_xyz(A):
    B = get_B_list(A)
    P=[]
    length = len(B)
    for i in range(length):
        for j in range(length):
            if j > i:
                a = vector(B[i])
                b = vector(B[j])
                c = a.cross_product(b)
                if c.norm() !=0:
                    c = c / norm(c)
                    if c[2] >= 0:
                       P.append([c[0],c[1],c[2]])
                       if c[2] == 0:
                            P.append([-1*c[0],-1*c[1],c[2]])
                    else:
                        P.append([-1*c[0],-1*c[1],-1*c[2]])
    return P
##########################
def get_int_lines(A):
    B= get_B_list(A)
    P=[]
    length = len(B)
    for i in range(length):
       for j in range(length):
           if j > i:
               a = vector(B[i])
               b = vector(B[j])
               c = a.cross_product(b)
               if c.norm() !=0:
                   c = c / norm(c)
                   if c[2] >= 0:
```

```
                    P.append([i,j])
                    if c[2] == 0:
                         P.append([i,j])
                 else:
                    P.append([i,j])
return P
```

References

[1] O. Bastani, C. Hillar, D. Popov & J. M. Rojas, *Randomization, Sums of Squares, Near-Circuits, and Faster Real Root Counting*, Randomization, Relaxation, and Complexity in Polynomial Equation Solving, Contemporary Mathematics, vol. 556, pp. 145166, AMS Press, 2011.

[2] A. Dickenstein, J. M. Rojas, K. Rusek, J. Shih, *Extremal real algebraic geometry and A-discriminants*, Mosc. Math. J, vol. 7, 3, pp. 425 - 452

[3] I.M. Gelfand, M. Kapranov & A. V. Zelevinsky, *Discriminants, Resultants and Multidimensional Determinants*, Birkhäuser, Boston, 1994.

[4] Kapranov, Misha *A characterization of $\mathcal{A}$-discriminantal hypersurfaces in terms of the logarithmic Gauss map*, Mathematicsche Annalen, 290, 1991, pp. 277-285.

[5] P. Pbay, J. M. Rojas, David C Thompson, *Sturms Theorem with Endpoints*,

[6] K. Phillipson & M. Rojas *$\mathcal{A}$-Discriminants, and their Cuttings, for Complex Exponents*

[7] K. Rusek, *$\mathcal{A}$-Discriminant Varieties and Amoeba*, Thesis, Texas A&M University.

Texas A&M University - Corpus Christi
E-mail address: coronado_joann@yahoo.com

University of Puerto Rico, Río Piedras Campus
E-mail address: samuelperezupr@gmail.com

University of Hawaii at Hilo
E-mail address: bithiah@hawaii.edu