Gröbner Bases and the Neural Ideal

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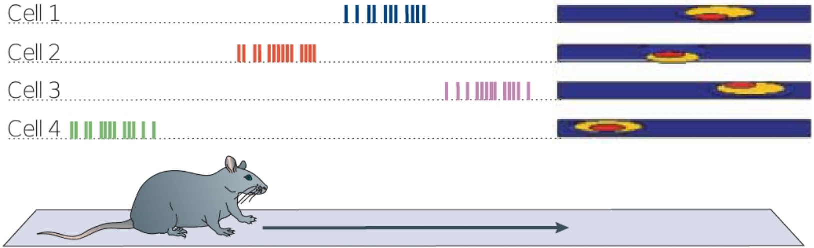
Biological Motivation

Place cells are neurons that encode spatial information.



We can describe the activity of n with binary strings of length n.

- 1 denotes a firing neuron
- 0 denotes a silent neuron

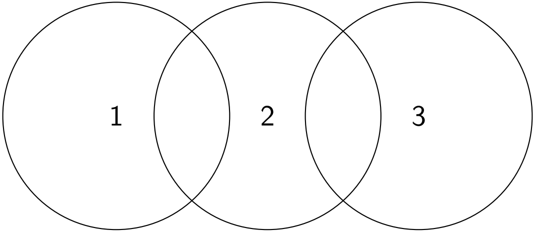
A neural code

Definition

Given a set of neurons labeled $\{1, \ldots, n\}$, a **neural code** on n neurons is a set of binary strings $C \subset \{0,1\}^n$.

Example

Let us consider the following code on 3 neurons:
$C = \{100, 110, 010, 011, 001\}$



Ideals and Varieties

Definition

Ideals: Let R be a commutative ring. A subset $I \subset R$ is an **ideal** of R if it has the following properties:

1. I is a subgroup of R under addition.
2. If $a \in I$, then $ra \in I$ for all $r \in R$.

An ideal I is said to be **generated** by a set A, and we write $I = \langle A \rangle$, if I is the set of all finite combinations of elements of A with coefficients in R.

Definition

Let $J \subset \mathbb{F}_2[x_1, \dots, x_n]$ be an ideal, and define the variety

$$V(J) = \{v \in \{0,1\}^n | f(v) = 0 \text{ for all } f \in J\}.$$

Pseudomonomials

Definition

For some $f \in \mathbb{F}_2[x_1, \ldots, x_n]$, f is a **pseudo-monomial** if f has the form

$$f = \prod_{i \in \sigma} x_i \prod_{j \in \tau} (1 + x_j) = x_\sigma \prod_{i \in \tau} (1 + x_i).$$

for some $\sigma, \tau \subset [n]$ with $\sigma \cap \tau = \emptyset$.

Example

Pseudo-monomial:

$$x_1 x_2 (1 + x_3)(1 + x_4)$$

Not pseudo-monomials:

$$x_1 x_2 + x_1 x_3 \text{ and } x_1^2 x_2$$

The neural ideal

Definition

For any $v \in \{0,1\}^n$, consider ρ_v, defined as

$$\rho_v = \prod_{i=1}^{n}(1 - v_i - x_i) = \prod_{\{i|v_i=1\}} x_i \prod_{\{j|v_j=0\}} (1 + x_j)$$

Definition

For a code C, the **neural ideal** $J_C = \langle \{\rho_v | v \notin C\} \rangle$.
Note: $V(J_C) = C$.

Definition

The **canonical form** of a neural ideal J_C is the set of all minimal pseudo-monomials that are elements of J_C.

Goal

The canonical form gives us a compact description of the relationships between receptive fields associated with a code.

The ultimate goal:
To find an efficient method to compute the canonical form of a neural code.

- Computing the canonical form is very computationally inefficient
- The computation is infeasible for codes on large numbers of neurons.
- (Petersen et al) Computing the Gröbner basis, another generating set of the ideal, is much faster.

Gröbner bases

Definition

A set $\{g_1, \ldots, g_t\} \subseteq I$ is a **Gröbner basis** of I if and only if the leading term of any element of I is divisible by one of the $\mathrm{LT}(g_i)$.

Definition (Criteria for a reduced Gröbner basis)

Let G be a Gröbner basis. G is a **reduced Gröbner basis** for all $g \in G$, no trailing term of any $g \in G$ is divisible by the leading term of any element of G.
Note: For a given monomial order the reduced Gröbner basisis unique.

Definition

Let I be an ideal. The **universal Gröbner basis** is the union of all the reduced Gröbner bases of I w.r.t. any monomial order.

Theorem (L.)

Let f be an pseudo-monomial, and let $G = \{g_1, \dots, g_k\}$ be a set of pseudo-monomials. If the remainder on division of f by $G = \{g_1, \dots, g_s\}$ is 0 for any monomial ordering, then for some $g \in G$, g divides f.

Pseudo-monomials

Proposition

Let $f = x_\sigma \prod_{i\in\tau}(1 + x_i)$ be a pseudo-monomial. Then we can write f as

$$f = \sum_{\gamma\in P(\tau)} x_\sigma x_\gamma,$$

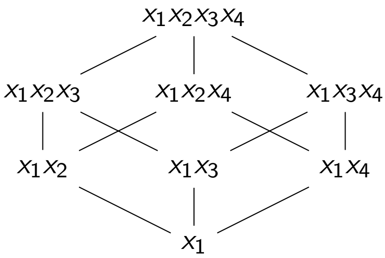
where $P(\tau)$ is the powerset of τ.

Notice that each term of f corresponds to an element of $P(\tau)$.

Hypercube of f

Example

Let $f = x_1(1+x_2)(1+x_3)(1+x_4)$. In this case, $\sigma = \{1\}$ and $\tau = \{2,3,4\}$.



hypercube of f

Pseudo-monomial divisibility

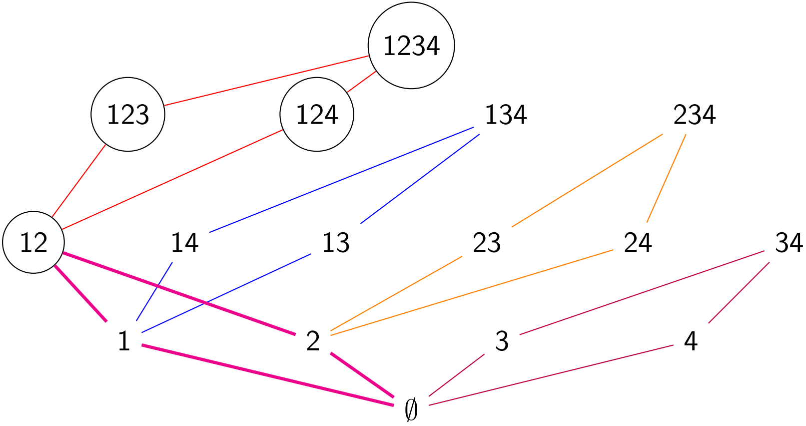
Lemma

Let $J \in \mathbb{F}_2[x_1, \ldots, x_n]$ be an ideal, and let f, g be pseudo-monomials such that $g = x_\alpha \prod_{i\in\beta}(1 + x_i)$ and $f = x_\sigma \prod_{j\in\tau}(1 + x_j)$. Then $g|f$ if and only if $\alpha \subset \sigma$ and $\beta \subset \tau$.

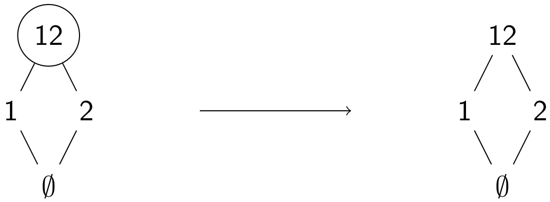
Lemma

Let $f = x_\sigma \prod_{i\in\tau}(1 + x_i)$ and let H be the hypercube of $P(\sigma \cup \tau)$. A pseudo-monomial h divides f if and only if the hypercube of h is a sub-cube of H and the hypercube of h intersects the Hasse diagram of $P(\sigma)$ at a unique vertex.

Geometric Intuition



Proof Idea



Initial State

End state

Theorem (L.)

Let C be a code, and J_C be the neural ideal of C. If the canonical form of J_C is a Gröbner basis, then the canonical form of J_C is a reduced Gröbner basis.

Theorem (L.)

Let J_C be a neural ideal and let G be the universal Gröbner basis of J_C. For all $g \in G$, if g is a pseudo-monomial, then g is in the canonical form of J_C.

Application

Compute Gröbner basis of neural ideal

↓

If all $g \in$ GB are pseudo-monomials, return GB

↓

Improves runtime when CF $=$ GB

Complementary Codes

Definition

Let $c \in \{0,1\}^n$ be a code. The **complement** of c is the code $c' \in \{0,1\}^n$ such that $c'_i = 1$ if and only if $c_i = 0$.

Definition

Let f be a pseudo-monomial such that $f = x_\sigma \prod_{i \in \tau}(1 + x_i)$. The **complement** of f, denoted f', is $f' = x_\tau \prod_{j \in \sigma}(1 + x_j)$.

Definition

A code $C \subset \{0,1\}^n$ is called **complement-complete** if for all $c \in C$, $c' \in C$ as well.

Complement-Complete implies CF $\neq$ GB

Theorem (L.)

Let C be a code on n neurons such that $C \subsetneq \{0,1\}^n$. If C is complement-complete, then the canonical form of J_C is not a Gröbner basis.

Example

$$\text{Let } C = \{111, 000, 110, 001, 100, 011\}.$$
$$C' = \{010, 101\}$$
$$J_C = \langle x_2(1+x_1)(1+x_3), x_1x_3(1+x_2)\rangle$$

Future work

What we know now: If an element of the Gröbner basis is a pseudo-monomial, then it is in the canonical form.

What we want to know: If an element of the Gröbner basis is *not* a pseudo-monomial, can we still use it to find elements of the canonical form?

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Thank you!