Classification of Unitarizable Representations of B_5

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Introduction to Representations

Definition: Representation

A **representation** of dimension n is a homomorphism from a group G into invertible matrices of size n. In notation that is a **representation** is a map $\varphi : G \to GL_n(K)$

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Definition: Irreducible Representation

A representation φ is called **irreducible** if the only G-invariant subspaces are trivial.

Unitarizable representations

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A representation V is said to be unitary if V is equipped with a Hermitian inner product such that for all $g \in G$ we have that $\langle \varphi(g)v|\varphi(g)w\rangle = \langle v|w\rangle$. A representation is called unitarizable if it can be equipped with such a Hermitian inner product.

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- A representation is unitary if it maps each group element to a unitary matrix. Or in finitely generated group if it maps each generators to a unitary matrix.
- We are studying the unitarizable representations of the braid group because these are important to topological quantum computing.

A Detour to Applications

What is a Quantum Computer

A quantum computer is an analogue of a regular computer that manipulates quantum bits. A quantum bit (or qbit) is the fundamental unit of quantum information.

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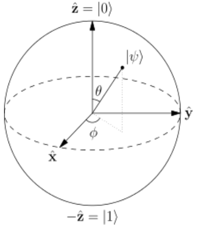

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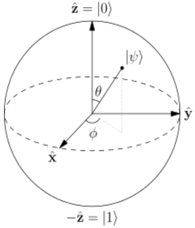

How to Perform Computation in a QC

In a quantum computer the logic gates are unitary transformations of the quantum state of each quibit. So in other words they are unitary matrices.

Topological Quantum Computation

What a TQC is

In a topological quantum computer the quantum state is on two dimensional quantum particles called anayons. A Unitary local representation of the braiding of these anyons allows for computation.

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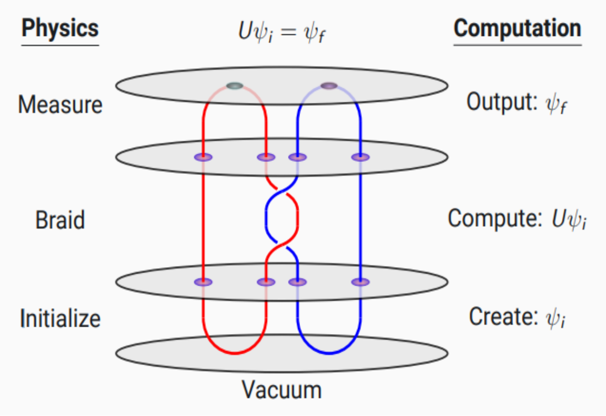

Useful Lemmas For Inner Products

Lemma

Let $\langle v|w\rangle_1$ be some Hermitian inner product on $\mathbb{C}^n$ then there exists some A such that $\langle v|w\rangle_1 = \langle v|w\rangle_A = \langle Av|w\rangle$. This matrix A has values $a_{ij} = \langle e_i|e_j\rangle_1$ where e_i and e_j are elements of the standard basis of $\mathbb{C}^n$.

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Define the adjoint operator * with respect to $\langle\cdot|\cdot\rangle_A$ as $U^* = A^{-1}U^\dagger A$ where $\dagger$ is the conjugate transpose. Then we have that $\langle Uv|Uw\rangle_A = \langle v|w\rangle_A$ for all $u, v \in \mathbb{C}^n$ if and only if $UU^* = I$.

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Equivelent Definition

Let (φ, V) be a representation over a complex vector space. Then assume that there is a φ_x such that $\varphi_x(b) = X^{-1}\varphi(b)X$. Then if φ_x is unitary with respect to $\langle u|v\rangle_1$ then φ is unitarizable.

The Braid Group

- Informally the braid group can be thought of as a group composed of the crossing of strings where braids which are isotopic are identified.
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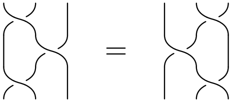

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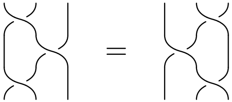

Definition: The Braid group

The braid group B_n is generated by the following
$\langle \sigma_1, \sigma_2, \cdots \sigma_{n-1} \mid \sigma_{i-1}\sigma_i\sigma_{i-1} = \sigma_i\sigma_{i-1}\sigma_i \textit{ and } \sigma_i\sigma_j = \sigma_j\sigma_i \textit{ if } |i-j| \geq 2 \rangle$

Known Representations of B_5

- The Burau representation is a well known representation which is unfortunately never irreducible.
- However the Burau Representation can be decomposed into the reduced Burau Representation and a one dimensional representation.

The (Reduced) Burau Representation

$$\beta(\sigma_1) = \left[\begin{array}{cc|c} -t & 1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & I_{n-3} \end{array}\right] \quad \beta(\sigma_i) = \left[\begin{array}{c|ccc|c} I_{i-2} & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & t & -t & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & I_{n-i-2} \end{array}\right]$$

$$\beta(\sigma_{n-1}) = \left[\begin{array}{c|cc} I_{n-3} & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & t & -t \end{array}\right]$$

Classification of the Representations of B_5

- Previous papers have classified all irreducible representations of B_5 of dimension less than five.
- They use representations built using Hecke Algebras denoted μ and $\hat{\mu}$.

Classification of Irreducible Representations by Dimension

They are listed by dimension.

1. There is just $\chi(y) : B_5 \to \mathbb{C}$ which is a constant mapping.
2. There are no irreducible representations.
3. The irreducible representations are all of the form $\chi(y) \otimes \hat{\beta}(z)$.
4. The irreducible representations are of the form $\chi(y) \otimes \beta(z)$ and $\chi(y) \otimes \hat{\mu}(z)$.
5. They are all equivalent to $\chi(y) \otimes \mu(z)$ or a tensor product of the standard representation

Unitarisablity of the Burau Representation

$$P_{n-1} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & s & & \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & s^{n-1} \end{bmatrix} J_{n-1} = \begin{bmatrix} s+s^{-1} & -1 & \dots & 0 \\ -1 & s+s^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ 0 & \dots & -1 & s^{n-1} \end{bmatrix}$$

Conjugating the Reduced Burau Representation

We have that $\beta(z)_S = P_{n-1}^{-1}\beta(z)P_{n-1}$ is unitary with respect to J_{n-1} as this was proved in a paper Squier.

- This implies that the reduced Burau representation is unitary when $J_{n-1} = X^*X$.

The Standard Representation

The Standard Representation

Define the representation $s(y) : B_n \to (C)^n$. By

$$\sigma_i = \begin{bmatrix} I_{i-1} & & & \\ & 0 & t & \\ & 1 & 0 & \\ & & & I_{n-i-1} \end{bmatrix}$$

Theorem

The Standard Representation is unitarizable if and only if t is on the unit circle.

The Standard Representation Continued

Proof of the Previous Theorem

We have the following by direct computation

$$s(t)(\sigma_i)(s(t)(\sigma_i))^\dagger = \begin{bmatrix} I_i & & \\ & t\bar{t} & \\ & & I_{n-i-1} \end{bmatrix}$$

Then clearly if t is on the unit circle we have that this is the identity so each matrix mapped to by the generators is unitary. For the other direction there is a considerable about of computation which will be in the appendix of my paper.

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- This representation is very useful in the classification of the representations of braid groups on n strands. In fact Inna Sysoeva proved that for $n \geq 9$ the standard representation is the only irreducible n dimensional representation up to tensor product.

How we approach finding more Unitary Conditions

A Tedious System of Equations

Let the representation $\beta(z)$ be unitary with respect to inner product $\langle \cdot | \cdot \rangle_A$. Since we have that $\varphi(z)(g)\varphi(z)(g)^* = I$ we have the equation $\varphi(z)(g)^\dagger A - A\varphi(z)(g)^{-1} = 0$.

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- We use the following pseudo code

```
A = symbolicMatrix(n)
for i = 1:4
E1 = B_i'*A - A*inv(B_i) == 0
V1 = eqnToMatrix(E1)
end
V = [ V1; V2; V3; V4 ]
```

Condtions on the Other Representations

- Using the code on the previous pages we can get conditions on the unitarity of the remaining representations of B_5

Theorem

The Hecke representations $\mu(z) : B_5 \to \mathbb{C}^5$, $\hat{\mu}(z) : B_5 \to \mathbb{C}^4$, and specialized Burau representation $\hat{\beta}(z) : B_5 \to \mathbb{C}^3$ are never unitarizable.

- These follow from computations to determine conditions on the entries of A.
- In each case if the representation were unitarizable this would imply that A has an all zero row, contradicting its inevitability.

Dealing with the Tensor Product

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Theorem

Given a representation $\chi(z) : B_5 \to \mathbb{C}^*$ which is defined as $\chi(z)(\sigma_i) = z$. Then $\chi(z) \otimes \varphi$ is unitarizable if and only if there exists an A such that $\langle \varphi(g)v | \varphi(g)w \rangle_A = c\langle v|w\rangle_A$ for all $v, w \in \mathbb{C}^n$ for some positive real c.

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Proof : If there exists an A such that $\langle \varphi(g)v | \varphi(g)w \rangle_A = c\langle v|w\rangle_A$ for all $v, w \in \mathbb{C}^n$ for some positive real c, then pick your favorite z such that $|z| = \frac{1}{\sqrt{c}}$. Now $\langle \chi \otimes \varphi(g)v | \chi \otimes \varphi(g)w \rangle_A = \langle z * \varphi(g)v | z * \varphi(g)w \rangle_A = |z|^2(c\langle v|w\rangle_A) = \langle v|w\rangle_A$. So assume that $\chi(z) \otimes \varphi$ is unitarizable, then by similar computation $c = \frac{1}{|z|^2}$.

Main Theorem and Future Work

All Unitarizable Low Dimensional Representations of B_5

Listed by dimension:

1. The only irreducible unitary representation is $\chi(z)$ where $|z| = 1$.
2. No such irreducible representations
3. No such irreducible representations
4. The only irreducible unitary representation is the Burau type representation $\chi(z) \otimes \beta(t)$ when $|z| = 1$ and the previously described J_n matrix is positive definite.
5. The standard representation $\chi(z) \otimes s(t) : B_5 \to \mathbb{C}^5$ when $|t| = 1, |z| = 1$ are the only such representations.

- The classification of all irreducible representations $(d \leq n)$ of B_n is complete. We will test the representations of B_n for $n = 6, 7, 8$.As for $n \geq 9$ the only irreducible representation is the standard.

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