THE DISTRIBUTION OF SHORT ORBITS OF SINGULAR MODULI

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Abstract. We study the asymptotic distribution of weak Maass forms averaged over short orbits of Heegner points. Under a mild condition on the growth of the size of these orbits, we give an asymptotic formula with a power-saving error term for these averages. We apply our results to compute the limiting distribution of short orbits of singular moduli.

1. Introduction and statement of results

1.1. **Overview.** In this paper we study the asymptotic distribution of weak Maass forms averaged over short orbits of Heegner points.

To summarize our results, let $k \geq 0$ be an integer and $M^!_{-2k}(1)$ be the space of weakly holomorphic modular forms of weight $-2k$ and level one. Define the differential operator $\mathcal{D}^k$ by

$$\mathcal{D}^k f := \frac{1}{(4\pi)^k} R_{-2} R_{-4} \cdots R_{-2k} f, \quad k \geq 1,$$

and $\mathcal{D}^0 f = f$, where R_t is the Maass weight raising operator

$$R_t f := 2i \frac{\partial}{\partial z} + \frac{t}{y}, \quad t \in \mathbb{Z}.$$

The operator $\mathcal{D}^k$ maps $M^!_{-2k}(1)$ to the space of weight zero weak Maass forms of level one.

Let $d < -4$ be an odd fundamental discriminant and Λ_d be the set of Heegner points of discriminant d on the modular curve $X_0(1)$. The class group G_d acts simply transitively on Λ_d.

For each d, choose a subgroup $H_d < G_d$ and a Heegner point $\tau_{0,d} \in \Lambda_d$. Consider the H_d-orbit

$$H_d \cdot \tau_{0,d} = \{\tau_{0,d}^\sigma : \ \sigma \in H_d\}$$

and the corresponding average

$$\mathrm{Av}_{H_d}(\mathcal{D}^k f, \tau_{0,d}) := \frac{1}{|H_d|} \sum_{\sigma \in H_d} \mathcal{D}^k f(\tau_{0,d}^\sigma).$$

We will give an asymptotic formula with a power-saving error term for $\mathrm{Av}_{H_d}(\mathcal{D}^k f, \tau_{0,d})$ as $|d| \to \infty$ for sequences of subgroups (H_d) satisfying a mild growth condition; see Theorem 1.1 and Corollary 1.3. We then apply this result when $k = 0$ and $f = j$ is the modular j-function to compute the limiting distribution of averages of short orbits of singular moduli; see Corollary 1.4.

1.2. **Quadratic forms and Heegner points.** We fix the following setup concerning quadratic forms and Heegner points.

Let $d < -4$ be an odd fundamental discriminant and $\mathcal{Q}_d$ be the set of positive definite, primitive, integral binary quadratic forms

$$Q(X,Y) = [a_Q, b_Q, c_Q](X,Y) = a_Q X^2 + b_Q XY + c_Q Y^2$$

of discriminant $b_Q^2 - 4a_Q c_Q = d$. There is a (right) action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{Q}_d$ defined by

$$Q = [a_Q, b_Q, c_Q] \longmapsto Q\gamma = [a_Q^\gamma, b_Q^\gamma, c_Q^\gamma] \quad \text{for} \quad \gamma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

where

$$\begin{aligned} a_Q^\gamma &= a_Q\alpha^2 + b_Q\alpha\gamma + c_Q\gamma^2, \\ b_Q^\gamma &= 2a_Q\alpha\beta + b_Q(\alpha\delta + \beta\gamma) + 2c_Q\gamma\delta, \\ c_Q^\gamma &= a_Q\beta^2 + b_Q\beta\delta + c_Q\delta^2. \end{aligned}$$

The set $\mathcal{Q}_d/\mathrm{SL}_2(\mathbb{Z})$ is a finite abelian group with respect to Gauss's law of composition of forms. Let $G_d = \mathcal{Q}_d/\mathrm{SL}_2(\mathbb{Z})$ be the class group and $h(d) = |G_d|$ be the class number.

To each form $Q \in \mathcal{Q}_d$ we associate a Heegner point τ_Q which is the root of $Q(X,1)$ given by

$$\tau_Q = \frac{-b_Q + \sqrt{D}}{2a_Q} \in \mathbb{H}.$$

We write $x_Q := \mathrm{Re}(\tau_Q)$ and $y_Q := \mathrm{Im}(\tau_Q)$. The Heegner points τ_Q are compatible with the action of $\mathrm{SL}_2(\mathbb{Z})$ in the sense that if $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then

$$\gamma(\tau_Q) = \tau_{Q\gamma^{-1}}. \tag{1}$$

We define the set of Heegner points of discriminant d by

$$\Lambda_d := \{\tau_{[Q]} : \ [Q] \in \mathcal{Q}_d/\mathrm{SL}_2(\mathbb{Z})\}.$$

Given two forms $Q, Q' \in \mathcal{Q}_d$, let $Q \circ Q'$ denote their composition. The group G_d acts simply transitively on Λ_d by

$$[Q'] \cdot \tau_{[Q]} = \tau_{[Q \circ Q']}.$$

We will also denote this action by $\tau_{[Q]}^{[Q']}$.

Recall that a form $Q \in \mathcal{Q}_d$ is *reduced* if

$$|b_Q| \leq a_Q \leq c_Q,$$

and if, in addition, $|b_Q| = a_Q$ or $a_Q = c_Q$, then $b_Q \geq 0$. Each class in $\mathcal{Q}_d/\mathrm{SL}_2(\mathbb{Z})$ contains a unique reduced form. Let $\mathcal{Q}_d^{\mathrm{red}}$ denote the set of reduced forms representing the classes in $\mathcal{Q}_d/\mathrm{SL}_2(\mathbb{Z})$. If $Q \in \mathcal{Q}_d^{\mathrm{red}}$, then the corresponding Heegner point τ_Q lies in the standard fundamental domain $\mathcal{F}$ for $\mathrm{SL}_2(\mathbb{Z})$.

Finally, let $\widehat{G_d}$ be the group of characters $\chi : G_d \to S^1$ and $H_d^\perp < \widehat{G_d}$ be the subgroup of characters χ which restrict to the identity on a subgroup $H_d < G_d$.

1.3. **Bounds for L–functions.** Let χ be a character of G_d and χ_d be the quadratic Dirichlet character of conductor d. Let g be an arithmetically normalized Hecke-Maass form for $\mathrm{SL}_2(\mathbb{Z})$ with eigenvalue $\lambda_g = 1/4 + t_g^2$ and Θ_χ be the theta function of weight one and level $|d|$ associated to χ. Let $L(g \otimes \chi, s)$ be the Rankin-Selberg L-function of $g \otimes \theta_\chi$, $L(\chi, s)$ be the L–function of χ, $L(\chi_d, s)$ be the L–function of χ_d and $\zeta(s)$ be the Riemann zeta function. We assume bounds of the form

$$L(g \otimes \chi, 1/2) \ll_\epsilon \lambda_g^{B_1+\epsilon} |d|^{\delta_1+\epsilon}, \tag{2}$$

$$L(\chi, 1/2 + it) \ll_\epsilon (1/4 + t^2)^{B_2+\epsilon} |d|^{\delta_2+\epsilon}, \tag{3}$$

$$L(\chi_d, 1/2 + it) \ll_\epsilon (1/4 + t^2)^{B_3+\epsilon} |d|^{\delta_3+\epsilon}, \tag{4}$$

$$\zeta(1/2 + it) \ll_\epsilon (1/4 + t^2)^{B_4+\epsilon} \tag{5}$$

for some absolute constants $B_1, B_2, B_3, B_4 > 0$, $0 < \delta_1 < 1/2$ and $0 < \delta_2, \delta_3 < 1/4$.

By Harcos and Michel [6], the bound (2) holds for some sufficiently large $B_1 > 0$ and $\delta_1 = 1499/3000$. By Duke, Friedlander and Iwaniec [3], the bound (3) holds with $B_2 = 5$ and $\delta_2 = 1/4 - 1/23041$. By Young [13], the bound (4) holds with $B_3 = 1/12$ and $\delta_3 = 1/6$. By Bourgain [1], the bound (5) holds with $B_4 = 13/168$. We note that stronger bounds in either the spectral or conductor aspect may exist; we stated here bounds in which both B_i and δ_i are given explicitly, with the exception of (2), in which case an explicit B_1 has not yet been given (see Remark 1.2).

The Lindelöf Hypothesis implies that the bounds (2) – (5) hold with $B_1 = B_2 = B_3 = B_4 = \delta_1 = \delta_2 = \delta_3 = 0$.

1.4. **Main results.** The following is our main result.

Theorem 1.1. *Let $k \geq 0$ be an integer and $f \in M^!_{-2k}(1)$ be a weakly holomorphic modular form with Fourier expansion*

$$f(z) = \sum_{m=0}^{N_\infty} a(-m) q^{-m} + \sum_{m=1}^{\infty} a(m) q^m, \quad q := e(z) = e^{2\pi i z}.$$

For each d, choose a subgroup $H_d < G_d$ and a Heegner point $\tau_{0,d} = \tau_{[Q_{\tau_{0,d}}]} \in \Lambda_d$. There is an absolute constant $0 < \delta < 1/2$ given by (6) such that

$$\mathrm{Av}_{H_d}(\mathcal{D}^k f, \tau_{0,d}) = \frac{1}{h(d)} \sum_{\substack{Q \in \mathcal{Q}_d^{\mathrm{red}} \\ y_Q > \frac{2}{\sqrt{3}} + |d|^{-(1/2-\delta)}}} C(\tau_{0,d}, Q) \sum_{m=0}^{N_\infty} a(-m) c_k(m, y_Q)\, e(-m\tau_Q)$$
$$+ \frac{3}{\pi} \beta_k(f) + O_\epsilon\left(|H_d|^{-1} |d|^{\delta+\epsilon}\right)$$

as $|d| \to \infty$ where

$$C(\tau_{0,d}, Q) := \sum_{\chi \in H_d^\perp} \overline{\chi}([Q_{\tau_{0,d}}^{-1} \circ Q]),$$

$$c_k(m, y) := \sum_{j=0}^{k} \frac{(-1)^j (k+j)! m^{k-j}}{(4\pi y)^j j! (k-j)!},$$

and

$$\beta_k(f) := \int_{\mathrm{reg}} \mathcal{D}^k f(z) d\mu$$

is the regularized integral defined by (22)*. Assuming the Lindelöf Hypothesis, we have* $\delta = 9/20$*.*

Remark 1.2. Let B_1, B_2, B_3, B_4 and $\delta_1, \delta_2, \delta_3$ be as in the bounds (2) − (5). Then the constant $\delta = \delta(B_1, B_2, B_3, B_4, \delta_1, \delta_2, \delta_3, \epsilon)$ in Theorem 1.1 is given by

$$\delta = \begin{cases} \frac{1}{2} - \frac{1-2\delta_1}{4(2A+1)}, & \delta_1 \geq 2\delta_2 \text{ and } A'(1-2\delta_1) \leq (1-4\delta_3)(2A+1) \\ \frac{1}{2} - \frac{1-4\delta_2}{4(2A+1)}, & \delta_1 \leq 2\delta_2 \text{ and } A'(1-4\delta_2) \leq (1-4\delta_3)(2A+1) \\ \frac{1}{2} - \frac{1-4\delta_3}{4A'}, & A'(1-2\delta_1) \geq (1-4\delta_3)(2A+1) \text{ and} \\ & A'(1-4\delta_2) \geq (1-4\delta_3)(2A+1) \end{cases} \tag{6}$$

where

$$A := \lfloor \max\{B_1/2 + 1 + \epsilon, B_2 + 1/4 + 2\epsilon\} \rfloor + 1,$$
$$A' := \lfloor B_3 + B_4 + 3/2 + 3\epsilon \rfloor + 1.$$

In order to give a numerical value for δ, we need numerical values for the constants B_i, δ_i. In Section 1.3 we listed values for all of these constants except B_1. The work of Harcos and Michel [6] gives a polynomial dependence on the spectral parameter in the bound (2), which ensures the existence of some sufficiently large B_1. However, it seems difficult to produce a numerical value for B_1.

If we impose a mild growth condition on the sequence of subgroups (H_d) then we can ensure a power-saving exponent in the error term of Theorem 1.1. In particular, this allows us to compute the limiting distribution of $\mathrm{Av}_{H_d}(\mathcal{D}^k f, \tau_{0,d})$ as $|d| \to \infty$.

Corollary 1.3. *Let* (H_d) *be a sequence of subgroups such that* $|H_d| \gg |d|^\eta$ *for some* $\eta > \delta$*. Then*

$$\mathrm{Av}_{H_d}(\mathcal{D}^k f, \tau_{0,d}) - \frac{1}{h(d)} \sum_{\substack{Q \in \mathcal{Q}_d^{\mathrm{red}} \\ y_Q > \frac{2}{\sqrt{3}} + |d|^{-(1/2-\delta)}}} C(\tau_{0,d}, Q) \sum_{m=0}^{N_\infty} a(-m) c_k(m, y_Q)\, e(-m\tau_Q)$$
$$= \frac{3}{\pi}\beta_k(f) + O_\epsilon(|d|^{-(\eta-\delta)+\epsilon})$$

as $|d| \to \infty$*.*

1.5. **Short orbits of singular moduli.** Let

$$j(z) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots$$

be the classical modular j-function. Given a Heegner point $\tau_{0,d} \in \Lambda_d$, the values

$$S_d := \{j(\tau_{0,d}^\sigma) : \ \sigma \in G_d\}$$

are algebraic numbers called *singular moduli*. These numbers are j-invariants of CM elliptic curves and generate the Hilbert class field of $K_d = \mathbb{Q}(\sqrt{d})$.

The class group G_d acts on the set of singular moduli S_d, and this action is equivariant in the sense that $j(\tau_{0,d})^\sigma = j(\tau_{0,d}^\sigma)$ for $\sigma \in G_d$. In particular,

$$\mathrm{Av}_{H_d}(j, \tau_{0,d}) = \frac{1}{|H_d|} \sum_{\sigma \in H_d} j(\tau_{0,d})^\sigma$$

is the average of the H_d-orbit of the singular modulus $j(\tau_{0,d})$.

In [4], Duke determined the limiting distribution of traces of singular moduli, proving that

$$\frac{1}{h(d)} \Big(\sum_{Q \in \mathcal{Q}_d^{\mathrm{red}}} j(\tau_Q) - \sum_{\substack{Q \in \mathcal{Q}_d^{\mathrm{red}} \\ y_Q > 1}} e(-\tau_Q) \Big) \longrightarrow 720$$

as $|d| \to \infty$. In particular, this resolved a conjecture of Bruinier, Jenkins and Ono [2] regarding the convergence of a Rademacher-type series expression for traces of singular moduli.

Since $\beta_0(j) = (\pi/3)720$ (see (24)), we immediately get the following special case of Corollary 1.3 which gives the limiting distribution of averages of short orbits of singular moduli.

Corollary 1.4. *Let (H_d) be a sequence of subgroups such that $|H_d| \gg |d|^\eta$ for some $\eta > \delta$. Then*

$$\frac{1}{|H_d|} \sum_{\sigma \in H_d} j(\tau_{0,d})^\sigma - \frac{1}{h(d)} \sum_{\substack{Q \in \mathcal{Q}_d^{\mathrm{red}} \\ y_Q > \frac{2}{\sqrt{3}} + |d|^{-(1/2-\delta)}}} C(\tau_{0,d}, Q) e\left(-\tau_Q\right) = 720 + O_\epsilon(|d|^{-(\eta-\delta)+\epsilon})$$

as $|d| \to \infty$.

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2. From averages on short orbits to twisted traces

Let G be a finite abelian group and $H < G$ be a subgroup. Let $\widehat{G}$ be the group of characters $\chi : G \to S^1$ and $H^\perp < \widehat{G}$ be the subgroup of characters χ which restrict to the identity on H. Given a function $f : G \to \mathbb{C}$, the Fourier transform $\hat{f} : \widehat{G} \to \mathbb{C}$ is defined by

$$\hat{f}(\chi) = \sum_{\sigma \in G} \overline{\chi}(\sigma) f(\sigma).$$

The Poisson summation formula states that

$$\frac{1}{|H|} \sum_{\sigma \in H} f(\sigma) = \frac{1}{|G|} \sum_{\chi \in H^\perp} \hat{f}(\chi).$$

Let $\phi : \mathbb{H} \to \mathbb{C}$ be an $\mathrm{SL}_2(\mathbb{Z})$-invariant function and $\tau \in \Lambda_d$ be a Heegner point. Define the evaluation map

$$e_{\phi,\tau} : G_d \to \mathbb{C}$$

by $e_{\phi,\tau}(\sigma) = \phi(\tau^\sigma)$. Then by the Poisson summation formula we have

$$\frac{1}{|H_d|}\sum_{\sigma\in H_d} e_{\phi,\tau}(\sigma) = \frac{1}{|G_d|}\sum_{\chi\in H_d^\perp}\widehat{e_{\phi,\tau_{0,d}}}(\chi),$$

or equivalently,

$$\mathrm{Av}_{H_d}(\phi,\tau) = \frac{1}{|G_d|}\sum_{\chi\in H_d^\perp}\mathrm{Tr}_{\overline{\chi},d}(\phi,\tau), \tag{7}$$

where the twisted trace is defined by

$$\mathrm{Tr}_{\chi,d}(\phi,\tau) := \sum_{\sigma\in G_d}\chi(\sigma)\phi(\tau^\sigma).$$

Lemma 2.1. *Let $\tau = \tau_{[Q_\tau]} \in \Lambda_d$. Then*

$$\mathrm{Tr}_{\chi,d}(\phi,\tau) = \chi([Q_\tau])^{-1}\sum_{Q\in\mathcal{Q}_d^{\mathrm{red}}}\chi(Q)\phi(\tau_Q).$$

Proof. Write

$$\mathcal{Q}_d/\mathrm{SL}_2(\mathbb{Z}) = \{[Q_1],\ldots,[Q_{h(d)}]\}.$$

Set $\tau = \tau_{[Q_j]}$ for some fixed j. Then

$$\mathrm{Tr}_{\chi,d}(\phi,\tau) = \sum_{i=1}^{h(d)}\chi([Q_i])\phi(\tau_{[Q_j]}^{[Q_i]}) = \sum_{i=1}^{h(d)}\chi([Q_i])\phi(\tau_{[Q_i\circ Q_j]}).$$

Now, the form $Q_i\circ Q_j$ is $\mathrm{SL}_2(\mathbb{Z})$-equivalent to a unique reduced form Q_{ij}, and Q_i is $\mathrm{SL}_2(\mathbb{Z})$-equivalent to $Q_{ij}\circ Q_j^{-1}$. Hence

$$\begin{aligned}\sum_{i=1}^{h(d)}\chi([Q_i])\phi(\tau_{[Q_i\circ Q_j]}) &= \sum_{i=1}^{h(d)}\chi([Q_{ij}\circ Q_j^{-1}])\phi(\tau_{[Q_{ij}]})\\ &= \chi([Q_j])^{-1}\sum_{i=1}^{h(d)}\chi([Q_{ij}])\phi(\tau_{[Q_{ij}]})\\ &= \chi([Q_j])^{-1}\sum_{Q\in\mathcal{Q}_d^{\mathrm{red}}}\chi(Q)\phi(\tau_Q).\end{aligned}$$

□

3. Bounds for twisted traces of automorphic functions

Let $\mathcal{F}$ denote the standard fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$. Given two $\mathrm{SL}_2(\mathbb{Z})$-invariant functions $\phi_1,\phi_2 : \mathbb{H}\to\mathbb{C}$, we define the Petersson inner product by

$$\langle\phi_1,\phi_2\rangle := \int_{\mathcal{F}}\phi_1(z)\overline{\phi_2}(z)d\mu(z)$$

where

$$d\mu(z) := \frac{dxdy}{y^2}$$

is the hyperbolic measure. The corresponding L_2-norm is given by

$$||\phi||_2 := \sqrt{\langle \phi, \phi\rangle} = \left(\int_{\mathcal{F}} |\phi(z)|^2 d\mu(z)\right)^{1/2}.$$

Let $\mathcal{D}(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})$ denote the space of $\mathrm{SL}_2(\mathbb{Z})$-invariant functions $\phi : \mathbb{H} \to \mathbb{C}$ such that ϕ and $\Delta\phi$ are both smooth and bounded, where

$$\Delta := -y^2(\partial_x^2 + \partial_y^2)$$

is the hyperbolic Laplacian. For $A \in \mathbb{Z}^+$ we let Δ^A denote the composition of Δ with itself A-times.

Proposition 3.1. *Let $\phi \in \mathcal{D}(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})$ and $\tau \in \Lambda_d$. Then*

$$\mathrm{Tr}_{\chi,d}(\phi,\tau) = C(\chi)\frac{3}{\pi}\langle \phi, 1\rangle + O_\epsilon(||\Delta^A\phi||_2|d|^{\delta_1/2+1/4+\epsilon/2}) + O_\epsilon(||\Delta^A\phi||_2|d|^{\delta_2+1/4+\epsilon})$$

for any integer $A > \max\{B_1/2 + 1 + \epsilon, B_2 + 1/4 + 2\epsilon\}$ where

$$C(\chi) := \sum_{\sigma\in G_d} \chi(\sigma).$$

Proof. Let $\{u_j\}_{j=1}^\infty$ be an orthonormal basis of Maass cusps forms for $\mathrm{SL}_2(\mathbb{Z})$ with Δ-eigenvalues $\lambda_j = 1/4 + t_j^2$. Define the non-holomorphic Eisenstein series

$$E(z,s) := \sum_{\gamma\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})} \mathrm{Im}(\gamma z)^s, \quad \mathrm{Re}(s) > 1$$

which is an eigenfunction for Δ with eigenvalue $s(1-s)$. We have the spectral expansion (see e.g. [9, Theorem 15.5])

$$\phi(z) = \frac{\langle \phi, 1\rangle}{\mathrm{vol}(\mathcal{F})} + \sum_{j=1}^\infty \langle \phi, u_j\rangle u_j(z) + \frac{1}{4\pi}\int_{\mathbb{R}} \langle \phi, E(\cdot, 1/2+it)\rangle E(z, 1/2+it)dt$$

which converges pointwise absolutely and uniformly on compact subsets of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. Using $\mathrm{vol}(\mathcal{F}) = \pi/3$, this gives

$$\mathrm{Tr}_{\chi,d}(\phi,\tau) = C(\chi)\frac{3}{\pi}\langle \phi, 1\rangle + \sum_{j=1}^\infty \langle \phi, u_j\rangle W_{\chi,d,j} + \frac{1}{4\pi}\int_{\mathbb{R}} \langle \phi, E(\cdot, 1/2+it) W_{\chi,d}(t)dt \tag{8}$$

where

$$W_{\chi,d,j} := \sum_{\sigma\in G_d} \chi(\sigma)u_j(\tau^\sigma)$$

and

$$W_{\chi,d}(t) := \sum_{\sigma\in G_d} \chi(\sigma)E(\tau^\sigma, 1/2+it).$$

Now, by a formula of Waldspurger and Zhang [12, 14] (see also [6] and [11, Section 3]) we have

$$|W_{\chi,d,j}|^2 = \frac{\sqrt{|d|}}{2}\frac{L(\widetilde{u_j}\otimes\chi, 1/2)}{L(\mathrm{sym}^2\widetilde{u_j}, 1)}$$

where $\widetilde{u_j}$ is the arithmetically normalized Maass form corresponding to u_j. Then by the Hoffstein/Lockhart bound [7]

$$L(\mathrm{sym}^2\widetilde{u_j}, 1) \gg_\epsilon \lambda_j^{-\epsilon}$$

and the bound (2)

$$L(\widetilde{u_j} \otimes \chi, 1/2) \ll_\epsilon \lambda_j^{B_1+\epsilon}|d|^{\delta_1+\epsilon}$$

we get

$$W_{\chi,d,j} \ll_\epsilon \lambda_j^{\frac{B_1}{2}+\epsilon}|d|^{\delta_1/2+1/4+\epsilon/2}. \tag{9}$$

Similarly, by Gross/Zagier [5] we have

$$W_{\chi,d}(t) = 2^{-(1/2+it)}|d|^{(1/2+it)/2}\frac{L(\chi, 1/2+it)}{\zeta(1+2it)}. \tag{10}$$

Then by the bound (3)

$$L(\chi, 1/2+it) \ll_\epsilon (1/4+t^2)^{B_2+\epsilon}|d|^{\delta_2+\epsilon}$$

and the standard bound

$$\zeta(1+2it) \gg_\epsilon (1/4+t^2)^{-\epsilon} \tag{11}$$

we get

$$W_{\chi,d}(t) \ll_\epsilon (1/4+t^2)^{B_2+2\epsilon}|d|^{\delta_2+1/4+\epsilon}. \tag{12}$$

By a repeated application of Stokes' theorem (see e.g. [8, Lemma 4.1]), for any $A \in \mathbb{Z}^+$ we have

$$\langle \phi, u_j \rangle = \lambda_j^{-A}\langle \Delta^A\phi, u_j\rangle \tag{13}$$

and

$$\langle \phi, E(\cdot, 1/2+it)\rangle = (1/4+t^2)^{-A}\langle \Delta^A\phi, E(\cdot, 1/2+it)\rangle. \tag{14}$$

Also, Parseval's identity yields (see e.g. [9, (15.17)])

$$\sum_{j=1}^{\infty}|\langle \Delta^A\phi, u_j\rangle|^2 + \frac{1}{4\pi}\int_{\mathbb{R}}|\langle \Delta^A\phi, E(\cdot, 1/2+it)\rangle|^2dt = ||\Delta^A\phi||_2^2. \tag{15}$$

Finally, by Weyl's law

$$|\{t_j : \ |t_j| \leq T\}| \ll T^2$$

and the bound $\lambda_j \geq 1/4$, summation by parts shows that the series

$$\sum_{j=1}^{\infty}\frac{1}{\lambda_j^{A'}} \tag{16}$$

converges for any integer $A' > 2$.

Then (13), (14), the Cauchy-Schwarz inequality, (15) and (16) give

$$\sum_{j=1}^{\infty}\langle \phi, u_j\rangle\lambda_j^{B_1/2+\epsilon} \ll ||\Delta^A\phi||_2 \tag{17}$$

and

$$\int_{\mathbb{R}} \langle \phi, E(\cdot, 1/2+it)\rangle (1/4+t^2)^{B_2+2\epsilon} dt \ll ||\Delta^A \phi||_2 \tag{18}$$

for any $A > \max\{B_1/2+1+\epsilon, B_2+1/4+2\epsilon\}$.

The proposition now follows by combining (8), (9), (12), (17) and (18). □

4. Fourier expansion of $\mathcal{D}^k f$

Here we state the Fourier expansion of $\mathcal{D}^k f$. Recall that the Kloosterman sum is defined by

$$S(a,b;c) := \sum_{\substack{d \pmod{c} \\ (c,d)=1}} e\left(\frac{a\bar{d}+bd}{c}\right)$$

where $\bar{d}$ is the multiplicative inverse of $d \pmod{c}$. Also, let I_ν denote the I-Bessel function of order ν.

By [10, Propositions 5.3 and 6.2], we have the following Fourier expansion.

Proposition 4.1. *Let $k \geq 0$ be an integer and $f \in M^!_{-2k}(1)$. Then*

$$\mathcal{D}^k f(z) = \sum_{n=0}^{N_\infty} a(-n)e(-nz)c_k(n,y) + \sum_{n=1}^{\infty} B_k(n,y)e(nz),$$

where for $0 \leq n \leq N_\infty$

$$c_k(n,y) := \sum_{j=0}^{k} \frac{(-1)^j (k+j)! n^{k-j}}{(4\pi y)^j j!(k-j)!},$$

and for $n \geq 1$,

$$B_k(n,y) := \frac{2\pi}{\sqrt{n}} S_k(n) \sum_{j=0}^{k} \frac{(k+j)!}{(4\pi n y)^j j!(k-j)!}$$

with

$$S_k(n) := \sum_{m=1}^{N_\infty} a(-m) m^{k+1/2} \sum_{c>0} \frac{S(-m,n;c)}{c} I_{2k+1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

5. Regularization of $\mathcal{D}^k f$

In this section we recall the construction of a function which regularizes the function $\mathcal{D}^k f$ in the cusp at ∞.

Let $\phi_0 : \mathbb{R} \to [0,1]$ be a C^∞ function such that

$$\phi_0(t) := \begin{cases} 0 & \text{if } t \leq 0 \\ 1 & \text{if } t \geq 1. \end{cases} \tag{19}$$

Let $0 < \eta < 1$. Define the Poincaré series

$$f_{k,\eta}(z) := \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} g_{k,\eta}(\gamma z) \tag{20}$$

where

$$g_{k,\eta}(z) := \sum_{m=0}^{N_\infty} a(-m)\psi_{m,k,\eta}(\mathrm{Im}(z))e(-mz),$$

$$\psi_{m,k,\eta}(y) := \phi_0\left(\frac{y-2/\sqrt{3}}{\eta}\right)c_k(m,y).$$

Then define the regularized function

$$f_{k,\eta}^{\mathrm{reg}}(z) := \mathcal{D}^k f(z) - f_{k,\eta}(z). \tag{21}$$

By [10, Proposition 7.1] we have the following result.

Proposition 5.1. *For $y \geq 2/\sqrt{3}+\eta$ we have*

$$f_{k,\eta}^{\mathrm{reg}}(z) = \sum_{n=1}^{\infty} b_k\,(n,y)\,e\,(nz)$$

where $b_k(n,y) := B_k(n,y)$ if $k \geq 1$ and $b_k(n,y) := a(n)$ if $k=0$.

6. Regularized integrals

For a fixed $Y > 2/\sqrt{3}$, define the truncated fundamental domain

$$\mathcal{F}_Y := \{z \in \mathcal{F} : \mathrm{Im}(z) \leq Y\}.$$

Then if $f \in M_{-2k}^!(1)$ we define the regularized integral of $D^k f$ by

$$\beta_k(f) = \int_{\mathrm{reg}} \mathcal{D}^k f(z)d\mu := \lim_{Y\to\infty}\int_{\mathcal{F}_Y}\mathcal{D}^k f(z)d\mu. \tag{22}$$

By [10, Lemma 10.3], this limit always exists and

$$\langle f_{k,\eta}^{\mathrm{reg}}, 1\rangle = \beta_k(f). \tag{23}$$

Finally, by [10, Proposition 10.4] we have

$$\beta_0(f) = \frac{\pi}{3}\left(a(0) - 24\sum_{n=1}^{N_\infty} a(-n)\sigma_1(n)\right)$$

where $\sigma_1(n)$ is the sum of all positive divisors of n. In particular, if $f = j$ is the modular j-function, then

$$\beta_0(j) = \frac{\pi}{3}720. \tag{24}$$

7. Proof of Theorem 1.1

We will deduce Theorem 1.1 from the following result.

Proposition 7.1. *Let* $\tau_{0,d} = \tau_{[Q_{\tau_{0,d}}]} \in \Lambda_d$. *Then*

$$\begin{aligned}\mathrm{Tr}_{\chi,d}(\mathcal{D}^k f, \tau_{0,d}) = \chi([Q_{\tau_{0,d}}])^{-1} \sum_{\substack{Q \in \mathcal{Q}_d^{\mathrm{red}} \\ y_Q > \frac{2}{\sqrt{3}} + \eta}} \chi(Q) \sum_{m=0}^{N_\infty} a(-m) c_k(m, y_Q)\, e(-m\tau_Q) + C(\chi)\frac{3}{\pi}\beta_k(f) \\ + O_\epsilon(\eta^{-2A}|d|^{\delta_1/2+1/4+\epsilon}) + O_\epsilon(\eta^{-2A}|d|^{\delta_2+1/4+\epsilon/2}) \\ + O(\eta|d|^{1/2+\epsilon}) + O_\epsilon(\eta^{-(A'-1)}|d|^{\delta_3+1/4+\epsilon})\end{aligned}$$

for any integers $A, A' > 0$ *with* $A > \max\{B_1/2 + 1 + \epsilon, B_2 + 1/4 + 2\epsilon\}$ *and* $A' > B_3 + B_4 + 3/2 + 3\epsilon$.

Proof. By (21) we have

$$\mathrm{Tr}_{\chi,d}(\mathcal{D}^k f, \tau_{0,d}) = \mathrm{Tr}_{\chi,d}(f_{k,\eta}^{\mathrm{reg}}, \tau_{0,d}) + \mathrm{Tr}_{\chi,d}(f_{k,\eta}, \tau_{0,d}).$$

By Proposition 5.1 we have $f_{k,\eta}^{\mathrm{reg}} \in \mathcal{D}(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})$. Hence by Proposition 3.1 and (23) we get

$$\mathrm{Tr}_{\chi,d}(f_{k,\eta}^{\mathrm{reg}}, \tau_{0,d}) = C(\chi)\frac{3}{\pi}\beta_k(f) + O_\epsilon(\eta^{-2A}|d|^{\delta_1/2+1/4+\epsilon}) + O_\epsilon(\eta^{-2A}|d|^{\delta_2+1/4+\epsilon/2})$$

for any integer $A > \max\{B_1/2 + 1 + \epsilon, B_2 + 1/4 + 2\epsilon\}$.

By Lemma 2.1 we have

$$\mathrm{Tr}_{\chi,d}(f_{k,\eta}, \tau_{0,d}) = \chi([Q_{\tau_{0,d}}])^{-1} \sum_{Q \in \mathcal{Q}_d^{\mathrm{red}}} \chi(Q) f_{k,\eta}(\tau_Q).$$

A straightforward modification of [10, Lemma 9.1] yields the decomposition

$$\sum_{Q \in \mathcal{Q}_d^{\mathrm{red}}} \chi(Q) f_{k,\eta}(\tau_Q) = \mathrm{T}_{\chi,d,1} + \mathrm{T}_{\chi,d,2}$$

where

$$\mathrm{T}_{\chi,d,1} := \sum_{\substack{Q \in \mathcal{Q}_d^{\mathrm{red}} \\ y_Q > \frac{2}{\sqrt{3}} + \eta}} \chi(Q) \sum_{m=0}^{N_\infty} a(-m) c_k(m, y_Q)\, e(-m\tau_Q)$$

$$\mathrm{T}_{\chi,d,2} := \sum_{\substack{Q \in \mathcal{Q}_d^{\mathrm{red}} \\ \frac{\sqrt{2}}{3} < y_Q \le \frac{2}{\sqrt{3}} + \eta}} \chi(Q) \sum_{m=0}^{N_\infty} a(-m) c_k(m, y_Q)\, e(-m\tau_Q).$$

Since $|\chi([Q])| = 1$ for any $[Q] \in \mathcal{Q}_d/\mathrm{SL}_2(\mathbb{Z})$, an estimate gives

$$T_{\chi,d,2} \ll \Lambda(d, \eta)$$

where

$$\Lambda(d, \eta) := |\{Q \in \mathcal{Q}_d^{\mathrm{red}} :\ \sqrt{2}/3 < y_Q \le 2/\sqrt{3} + \eta\}|.$$

We next bound $\Lambda(d,\eta)$ along the lines of [10, Lemma 9.2]. Let $\phi_\eta : \mathbb{R} \to [0,1]$ be a smooth function which is supported on $(2/\sqrt{3}-\eta, 2/\sqrt{3}+2\eta)$, which equals 1 on $[2/\sqrt{3}, 2/\sqrt{3}+\eta]$, and which satisfies

$$\phi_\eta^{(\ell)} \ll \eta^{-\ell}, \quad \ell = 0,1,2,\ldots. \tag{25}$$

Define the Poincaré series

$$P_\eta(z) := \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \phi_\eta(\mathrm{Im}(\gamma z)).$$

Then by construction we get

$$|\Lambda(d,\eta)| \leq \sum_{Q \in \mathcal{Q}_d^{\mathrm{red}}} P_\eta(\tau_Q).$$

By [8, (7.12)] we have

$$P_\eta(z) = \frac{3}{\pi}\widehat{\phi_\eta}(1) + \frac{1}{2\pi i}\int_{\mathbb{R}} \widehat{\phi_\eta}(1/2+it)E(z,1/2+it)dt$$

where

$$\widehat{\phi_\eta}(s) := \int_0^\infty \phi_\eta(u)u^{-(s+1)}du.$$

Thus

$$\sum_{Q \in \mathcal{Q}_d^{\mathrm{red}}} P_\eta(\tau_Q) = \frac{3}{\pi}\widehat{\phi_\eta}(1)h(d) + \frac{1}{2\pi i}\int_{\mathbb{R}} \widehat{\phi_\eta}(1/2+it)W_{\mathbf{1},d}(t)dt.$$

An estimate gives

$$\widehat{\phi_\eta}(1) \ll \eta.$$

Further, by (10) and the factorization

$$\zeta_{\mathbb{Q}(\sqrt{d})}(s) = \zeta(s)L(\chi_d, s)$$

we have

$$W_{\mathbf{1},d}(t) = 2^{-(1/2+it)}|d|^{(1/2+it)/2}\frac{\zeta(1/2+it)}{\zeta(1+2it)}L(\chi_d, 1/2+it).$$

Hence the bounds (11), (5) and (4) yield

$$W_{\mathbf{1},d}(t) \ll_\epsilon (1/4+t^2)^{B_3+B_4+3\epsilon}|d|^{\delta_3+1/4+\epsilon}.$$

It follows that

$$\sum_{Q \in \mathcal{Q}_d^{\mathrm{red}}} P_\eta(\tau_Q) = O(\eta h(d)) + O_\epsilon(c(\eta)|d|^{\delta_3+1/4+\epsilon})$$

where

$$c(\eta) := \int_0^\infty |\widehat{\phi_\eta}(1/2+it)|(1/4+t^2)^{B_3+B_4+3\epsilon}dt.$$

Integrate by parts A'-times and use the bound (25) to obtain

$$\widehat{\phi_\eta}(1/2+it) \ll \frac{\eta^{-(A'-1)}}{(1/4+t^2)^{A'-1}}.$$

Hence

$$c(\eta) \ll \eta^{-(A'-1)}$$

for $A' > B_3 + B_4 + 3/2 + 3\epsilon$. We have shown

$$\Lambda(d,\eta) = O(\eta h(d)) + O_\epsilon(\eta^{-(A'-1)}|d|^{\delta_3+1/4+\epsilon}).$$

An inspection of [10, Lemma 9.3] gives

$$||\Delta^A f_{k,\eta}^{\mathrm{reg}}||_2 \ll \eta^{-2A}.$$

Further, we have the bound

$$h(d) \ll |d|^{1/2+\epsilon}.$$

Putting things together, we obtain the result.

□

Proof of Theorem 1.1. By Proposition 7.1, (7) and orthogonality, we have

$$\begin{aligned}\mathrm{Av}_{H_d}(\mathcal{D}^k f, \tau_{0,d}) = \frac{1}{h(d)} &\sum_{\substack{Q\in\mathcal{Q}_d^{\mathrm{red}} \\ y_Q > \frac{2}{\sqrt{3}}+\eta}} C(\tau_{0,d}, Q) \sum_{m=0}^{N_\infty} a(-m) c_k\left(m, y_Q\right) e\left(-m\tau_Q\right) + \frac{3}{\pi}\beta_k(f) \\ &+ E(\delta_1,\delta_2,\delta_3,A,A')\end{aligned}$$

where

$$C(\tau_{0,d}, Q) := \sum_{\chi\in H_d^\perp} \overline{\chi}([Q_{\tau_{0,d}}^{-1}\circ Q]),$$

and the error term is

$$\begin{aligned}E(\delta_1,\delta_2,\delta_3,A,A') := &O_\epsilon(|H_d|^{-1}\eta^{-2A}|d|^{\delta_1/2+1/4+\epsilon}) + O_\epsilon(|H_d|^{-1}\eta^{-2A}|d|^{\delta_2+1/4+\epsilon/2}) \\ &+ O_\epsilon(|H_d|^{-1}\eta|d|^{1/2+\epsilon}) + O_\epsilon(|H_d|^{-1}\eta^{-(A'-1)}|d|^{\delta_3+1/4+\epsilon})\end{aligned}$$

for any $A > \max\{B_1/2 + 1 + \epsilon, B_2 + 1/4 + 2\epsilon\}$ and $A' > B_3 + B_4 + 3/2 + 3\epsilon$.

Let $\eta = |d|^{-b}$, where b will be chosen to minimize the error term. We have

$$E(\delta_1,\delta_2,\delta_3,A,A') = O_\epsilon(|H_d|^{-1}|d|^{\delta+\epsilon}),$$

where

$$\delta := \delta(\delta_1,\delta_2,\delta_3,A,A') = \max\left\{\frac{\delta_1}{2} + \frac{1}{4} + 2Ab, \delta_2 + \frac{1}{4} + 2Ab, \frac{1}{2} - b, \delta_3 + \frac{1}{4} + (A'-1)b\right\}.$$

As b varies, δ is minimized when

$$b = b_0 := b(\delta_1,\delta_2,\delta_3,A,A') = \begin{cases} p_1, & p_1 \le p_2 \text{ and } p_1 \le p_3 \\ p_2, & p_2 \le p_1 \text{ and } p_2 \le p_3 \\ p_3, & p_3 \le p_1 \text{ and } p_3 \le p_2 \end{cases}$$

where

$$p_1 := \frac{1-2\delta_1}{4(2A+1)},$$
$$p_2 := \frac{1-4\delta_2}{4(2A+1)},$$
$$p_3 := \frac{1-4\delta_3}{4A'}.$$

Moreover, the minimal value of δ is $\frac{1}{2} - b_0$.

Setting $A = \lfloor \max\{B_1/2+1+\epsilon, B_2+1/4+2\epsilon\} \rfloor + 1$ and $A' = \lfloor B_3 + B_4 + 3/2 + 3\epsilon \rfloor + 1$ yields the result.

□

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