APPROXIMATING DISCRIMINANT VARIETIES OF ABRITRARY DEGREE UNIVARIATE TETRANOMIALS

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Abstract. We may solve sparse univariate tetranomials for their real positive roots using the discriminant variety, in time independent of the degree of the input polynomial. This useful technique is hampered by issues with determining sidedness, so we use instead approximations of the discriminant variety. We explore and implement improvements upon a piecewise linear approximation in order to develop an algorithm that outputs the number of real, positive zeroes of the inputted, arbitrary degree, polynomial.

1. introduction

Working with sparse polynomials and using their discriminant varieties offers us the possibility of solving over the field of real numbers very quickly, independent of degree. This approach encourages us to consider polynomials as described in the following definition:

Definition 1. *We call*

$$f = \sum_{i=1}^{n+k} c_i x^{a_i}$$

an n-variate $n+k$-nomial, with $f \in \mathbb{C}[x_1 ... x_n]$ and $c_i \neq 0$. The set $A = \{a_1 .... a_{n+k}\} \subset \mathbb{Z}^n$ is the support of the polynomial.

To understand the zero sets of these polynomials, we use the theory of A-discriminant varieties.

Definition 2. *Given an n-variate $n+k$-nomial, with support A, the* A-*discriminant variety is the closure of* $\nabla_{\mathrm{A}} = (c_1, ..., c_{n+k}) \in (\mathbb{C}^*)^{n+k}$, *where $f = c_1 x^{a_1} ... c_{n+k} x^{a_{n+k}}$ has a* degenerate root.

We call roots with multiplicity > 1 degenerate roots. The structure of the A discrinimant variety relates directly to the zero sets of the polynomials with the given support A, since degeneracy exists at points of change in zero set isotopy. When the codimension of $\nabla_A = 1$, the discriminant variety is the zero locus of irreducible polynomial, but this polynomial gets very complicated when the degree increases. Since we want to work independent of degree, we cannot use this discriminant

polynomial to gain information about the zeroes of input polynomials. So, we turn to a alternate method, Horn-Kapranov Uniformization, which neatly parameterizes the A-discriminant variety. We begin with our polynomial f, and derive a system of equations by setting both f and xf' equal to zero, thereby identifying points of degeneracy. From this system, we derive a matrix $\hat{A}$.
For example, with $f = c_0 + c_1x^{a_1} + c_2x^{a_2}$, we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & \mathrm{a}_1 & \mathrm{a}_2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1x^{a_1} \\ c_2x^{a_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \hat{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & \mathrm{a}_1 & \mathrm{a}_2 \end{bmatrix}$$

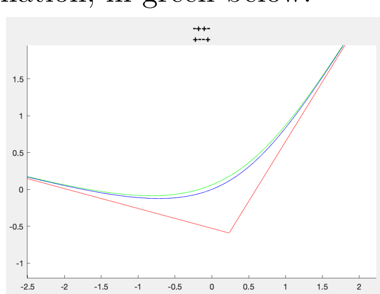
We then define another matrix B, whose columns form a basis of the right nullspace of $\hat{A}$.

Theorem 1. *The image of* $\Psi_{\mathrm{A},B}([\lambda]) = log|\lambda B^T|B$*, with* $[\lambda] \in \mathbb{P}_{\mathbb{R}}^{k-2}$*, is a slice of* $log|\nabla_{\mathrm{A}}|$*.*

Using this parameterization, with $k = 3$, we can visualize the discriminant variety in two dimensions and use its many properties. However, although HKU describes the discriminant variety more efficiently, it is not optimized for our goal of determining the chamber of the discriminant variety a given point in coefficient space is in. So, we turn to approximations: previous REU work developed both linear piece-wise and tropical amoeba approximations. Our work involves implementing the tropical amoeba approximation in such a way that allows for quickly determining sidedness of the given curve.

2. Techniques and Results

We seek an explicit description of the parametrically expressed tropical amoeba approximation, in green below.



We are looking at the *amoebae* of a family of trinomials of the form $x + y + 1 = 0$.

Definition 3. *The* Amoeba *of a polynomial is the coordinate-wise* $log|\bullet|$ *of its zero set.*

From this equation, we derive the curve $y = log(1 - e^{-x})$, and apply a simple change of variables which corresponds to rotation:

$$\begin{bmatrix} sin(m) & cos(m) \\ cos(n) & -sin(n) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

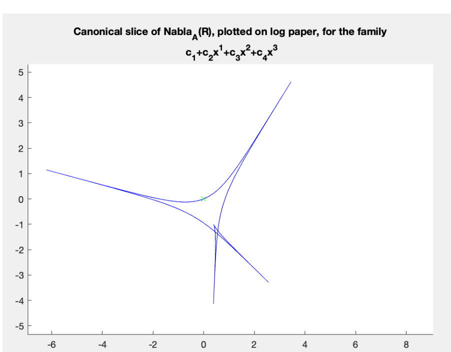
Our technique for finding the proper m and n is as follows. In each orthant, we have two 'rays' corresponding to rows of the B matrix. The points plotted in the orthant may be, for example, the image of the interval $(tan^{-1}(-B_{3,2}/B_{3,1}), tan^{-1}(-B_{2,2}/B_{2,1}))$ under the mapping $\psi_{A,B}$ from the previous page. When the vectors defined by these appropriate rows of the B matrix are *not* in quadrants I and IV or II and III, we simply define

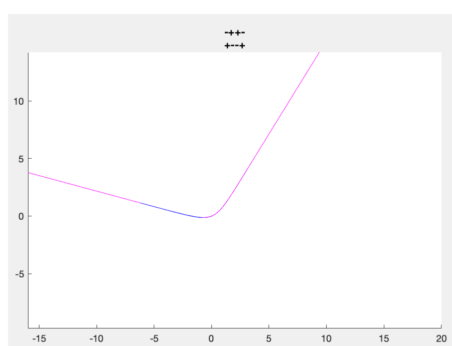
$$m = \pi - tan^{-1}(min) \tag{1}$$

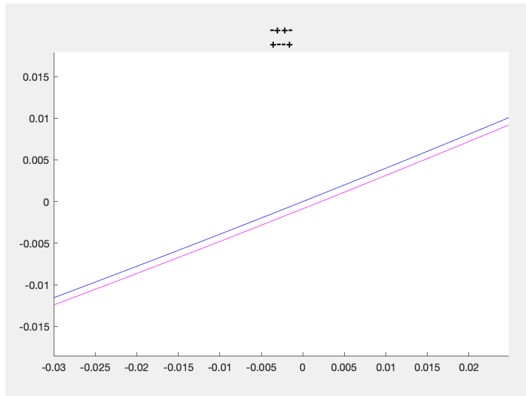
$$n = m + \pi/2 - |(tan^{-1}(min) - tan^{-1}(max)| \tag{2}$$

where min is the minimum of $-B_{a,2}/B_{a,1}$, $-B_{b,2}/B_{b,1}$, and max, respectively, is the maximum, with a and b being the two rows of B related to the specific orthant in which we are approximating.

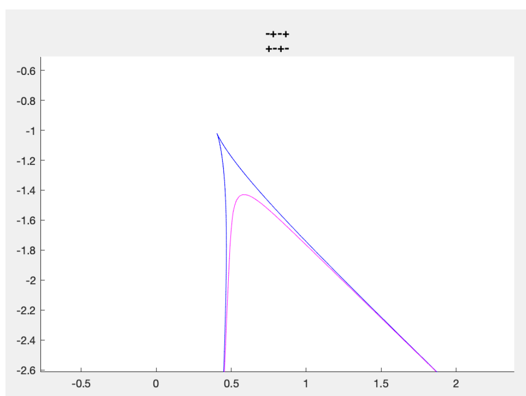
In the other case, where the vectors defined by the specified rows of the B matrix *are* in quadrants I and IV, or in II and III, we simply swap m and n in the above definitions. Lastly, when the angle formed by the rays, i.e. the angle facing outward in the plot of the discriminant variety shown below, is under 180°, we rotate by π. This produces the approximation shown below in magenta.







This approximation, of course, only suits those orthants without a cusp, a critical point in the map described by Horn Kapranov Uniformization. In these orthants, we need a different approximation.



Within 1 unit of the cusp, we turn to another family of curves: those defined by $y^3 = x^2$. The rotation we apply in this orthant depends only on one ray, rather than two as described on the previous page: the ray bisecting those in the orthant, defined by their sum. Then, we adjust the angle formed by the cusp through a constant a defined by a quadratic equation with input being the angle formed by the rays. Finally, the curve is bent to one side based on the distance between the point at which the rays, if extended, would intersect, and the bisecting ray mentioned above. This approximation produces curves such as the one below:

The magenta curve is used when the inputted point is outside the circle, and the red curve, given by the process detailed on the previous page, is used when inside.

Now, we have approximated the discriminant variety in every orthant in which it has points. What remains is to simply solve the system of equations given by derivatives of $\psi_{A,B}$ from page 2 to solve for the cusp. The program takes in a point in coefficient space, maps to the slice of logspace we are looking at, and evaluates the expression giving the approximation in the given orthant to determine sidedness. For example, an input of $\begin{bmatrix} -0.05 & 0.8 & -3 & 3 \end{bmatrix}$ produces the following output:

```
the given tetranomial has...
     3

positive real zeroes
-+-+
+-+-
...has corner = (0.51088,-1.28872)
   and rays (-0.02361,-0.54721) and (0.39213,-0.38240) !
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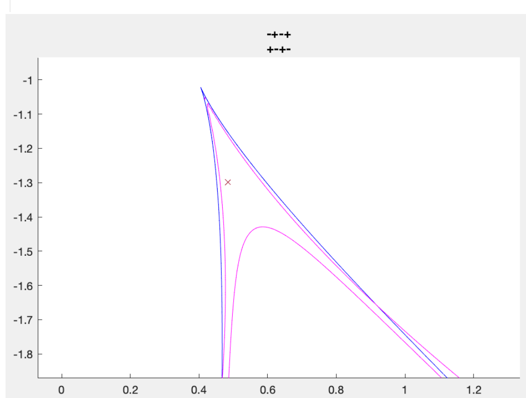

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3. Acknowledgements

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4. References

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