Homework 9 (Updated)

Math 302 (section 501), Fall 2016

This homework is due on Thursday, October 27.

0. *(This problem is not to be turned in.)*
   (a) Read Sections 2.5 and 5.1
   (b) Read about the Schröder-Bernstein Theorem and the continuum hypothesis (pages 174–175). Give two proofs that $|(0,1)| = |[0,1]|$, one using Schröder-Bernstein, and one that is direct.
   (c) (Practice Problems) Section 2.5 # 2, 6, 8, 12–15, 17–20, 22
   (d) (Practice Problems) Section 5.1 # 10, 12, 18, 24, 40, 43, 52, 56, 60
1. Section 2.5 # 4, 10, 18
2. Section 5.1 # 6, 8, 20, 32
3. Prove or disprove the following claims pertaining to sets A, B, and C:
   (a) If $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$.
   (b) If $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.
   (c) If $|A| \leq |B|$ and $|C| \leq |B|$, then $|A| = |C|$.
4. Prove or disprove:
   The set $\{x \in \mathbb{R} \mid x \text{ is rational or of the form } p \cdot \sqrt{5} \text{ with } p \text{ prime}\}$ *is countable.*
5. If n lines are drawn in a plane and no two lines are parallel, into how many regions (maximum number) do the lines separate the plane? Give a proof.
6. Prove the following by induction:
   (a) Let $\{f_j\}$ denote the Fibonacci numbers. For all $n \in \mathbb{Z}^+$, the following holds:
   $$f_1 + f_3 + \cdots + f_{2n-1} = f_{2n} .$$
   (b) For all $n \in \mathbb{Z}^+$,
   $$n^5 - n \text{ is divisible by } 5.$$
   You may use that $(k+1)^5 = k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1$.

7. A *string* is a finite sequence $a_1a_2\dots a_n$, where each a_i is an element of some set (this set is called an *alphabet*). For a string w, let $|w|$ denote its length and for every $1 \leq i, j \leq |w|$, let $w(i,j)$ denote the substring given by the i-th, $(i+1)$-st,...,j-th character in w. For two strings w_1 and w_2, we denote by $w_1 \cdot w_2$ its concatenation.

Examine the following pseudocode:

Input: A string w.
Output: A string w'.
begin
  if $|w| = 1$ **then**
    return w
  else
    return $\texttt{StringAlgo}\left(w\left(\left\lceil\frac{|w|+1}{2}\right\rceil, |w|\right)\right) \cdot \texttt{StringAlgo}\left(w\left(1, \left\lfloor\frac{|w|+1}{2}\right\rfloor\right)\right)$

Algorithm 1: `StringAlgo`

(a) Compute *StringAlgo("TAMU")*, *StringAlgo("Rabbit")* and *StringAlgo("Halloween")*. Describe in your own words, what the algorithm does.

(b) Use strong induction to prove: *StringAlgo* is injective.