Homework 11

Math 415 (section 502), Fall 2015

This homework is due on Thursday, November 12. You may cite results from class, as appropriate.

0. *(This problem is not to be turned in.)*
   (a) Read Section 18.
   (b) Section 18 # 38, 44, 55
1. True/false (No proofs necessary for this problem.)
   (a) S_3 is a simple group.
   (b) A_3 is a simple group.
   (c) A_5 is a simple group.
   (d) If $f : X \to Y$ is any function, and $U \subset X$, then $f^{-1}[f[X]] = X$.
   (e) If $f : X \to Y$ is any function, and $V \subset Y$, then $f[f^{-1}[V]] = V$.
   (f) If $f : X \to Y$ is any function, and $V \subset Y$, and f is onto (surjective), then $f[f^{-1}[V]] = V$.
2. Let R be a ring with unity, and let U denote the set of all units in R. Prove that U is closed under the operation of multiplication, and furthermore that it is a group (under multiplication).
3. Describe all ring homomorphisms $\mathbb{Z} \to \mathbb{Z}$.
4. Up to isomorphism, are there any finite simple abelian groups besides $\mathbb{Z}_1$ and $\mathbb{Z}_p$ (where p is a prime number)? Give a proof.
5. *The aim of this problem is to prove the following: if M is a proper normal subgroup of G, and G/M is simple, then M is a maximal normal subgroup of G.* Accordingly, assume that M is a proper normal subgroup of G, and G/M is simple.
   (a) Let $\gamma : G \to G/M$ be the usual homomorphism given by $x \mapsto xM$. Prove that if a normal subgroup N of G satisfies $M \subseteq N \subseteq G$, then $\gamma[N] \trianglelefteq G/M$. (*Hint*: use a theorem that tells us about the image of a normal subgroup.)
   (b) Use (a) to prove that either $\{M\} = \gamma[N]$ or $\gamma[N] = G/M$. (*Hint*: G/M is simple.)
   (c) Use (b) to prove that either $M = N$ or $N = G$. (*Hint*: the containments $\subseteq$ are by assumption; for $\supseteq$, use (b), but the true/false problems are supposed to warn you that you can not state that $\gamma^{-1}[\gamma[N]] = N$ without proof).
6. (a) Draw the lattice of the following subgroups of $\mathbb{Z}$: $\{0\}$, $5\mathbb{Z}$, $8\mathbb{Z}$, $9\mathbb{Z}$, and *all* subgroups that contain 12. Highlight or circle the subgroups that contain 12.
   (b) Draw the lattice of all subgroups of $\mathbb{Z}/12\mathbb{Z}$.
   (c) Use (a) and (b) to state a conjecture: *for a group G with a normal subgroup N, then $K \mapsto K/N$ defines a bijection between the set of subgroups of G that contain N and the set of* ______________________________.
   (d) (Challenge problem – optional!) Prove the conjecture you stated in (c).
7. Section 18 # 6, 12, 27, 33, 41