Homework 14

Math 415 (section 502), Fall 2015

This homework is due on Thursday, December 3. You may cite results from class or previous homeworks/exams.

0. *(This problem is not to be turned in.)* Read Section 21.
1. True/false (No proofs necessary for this problem.)
   (a) If D and D' are isomorphic integral domains, then their fields of quotients are isomorphic.
   (b) If p is a prime number, then $\mathbb{Z}_p$ is its own field of quotients.
   (c) If n is an integer that is *not* a prime number, then the field of quotients of $\mathbb{Z}_n$ is *not* defined.
   (d) The following is a field of fractions for $\mathbb{Q}[x]$:
   $$\mathbb{Q}(x) \ := \ \left\{ \frac{p}{q} \mid p, q \in \mathbb{Q}[x], \text{and } q \neq 0 \right\}$$
   (this field is called the *field of rational functions* with coefficients in $\mathbb{Q}$).
   (e) $\mathbb{Q}(x)$ is a field of fractions for $\mathbb{Z}[x]$.
   (f) $\mathbb{Z}_6 \times \mathbb{Z}_{10} \cong \mathbb{Z}_4 \times \mathbb{Z}_{15}$
2. A *zero* or *root* of a polynomial f is an element α for which $f(\alpha) = 0$. Find *all* zeroes of $x^{199} + x^{73} + x^{60} + x^6$ in $\mathbb{Z}_5$.
3. (a) Show (by counterexample) that the polynomial ring $\mathbb{Z}_{25}[x]$ is *not* an integral domain.
   (b) Prove that if D is an integral domain, then $D[x]$ is an integral domain.
4. (a) Prove that $D := \{a + bi \mid a, b \in \mathbb{Z}\}$ is a subring of the field $\mathbb{C}$. Is D an integral domain? Is it a field? Explain.
   (b) Explain why there exists a subfield of $\mathbb{C}$ that is a field of fractions of D.
   (c) Describe (as a subfield of $\mathbb{C}$) this field of fractions of D from part (b).
5. Does there exist a subfield of $\mathbb{Q}$ that is a field of fractions of the ring $\mathbb{Z}[x]$? If not, then give a proof. If so, explain, and describe this subfield.
6. (a) Section 21 #12
   (b) Use (a) to explain why the *partial ring of quotients* $Q(2\mathbb{Z}, 2\mathbb{Z} \setminus \{0\})$ is defined.
   (c) Does there exist a subfield of $\mathbb{Q}$ that is isomorphic to $Q(2\mathbb{Z}, 2\mathbb{Z} \setminus \{0\})$? If not, then give a proof. If so, explain, and describe this subring. Is it a field?
7. Section 21 #2, 4, 11
8. (Challenge – not to be turned in) Section 21 #5, 13–16