Homework 4

Math 415 (section 502), Fall 2015

This homework is due on Thursday, September 24. You may cite results from class, as appropriate. You may also cite facts from Calculus or Linear Algebra.

0. *(This problem is not to be turned in.)*
   (a) Read Sections 6–7.
   (b) Complete the following claim, and give a proof: *a function $\phi : A \to B$ is a bijection if and only if there exists a function $\psi : B \to A$ such that $\phi \circ \psi$ is the identity function on ____________ and $\psi \circ \phi$ is the identity function on ____________.*
   (c) Section 7 # 4, 6

1. Let G and G' be groups. Is the constant function $\phi : G \to G'$ given by $g \mapsto e_{G'}$ a homomorphism? (Here $e_{G'}$ is the identity element of G'.) Give a proof.

2. (No proofs necessary for this problem.)
   (a) Determine the subgroup $\langle 16, 28 \rangle$ of $\mathbb{Z}$.
   (b) Determine the subgroup $\langle 16, 28 \rangle$ of $\mathbb{Z}_{31}$.

3. (a) Prove that a homomorphism $\phi : \langle a \rangle \to G$ from a cyclic group generated by a to a group G is uniquely determined by $\phi(a)$. (*Hint*: What is $\phi(a^n)$?)
   (b) List all homomorphisms $\mathbb{Z} \to \mathbb{Z}_4$. (No proof necessary.)
   (c) Let G be a group. Prove that the set of homomorphisms $\mathbb{Z} \to G$ has the same cardinality as G.

4. (a) Let G be a group. Prove that a function $\phi : \mathbb{Z}_n \to G$ is a homomorphism if and only if $\phi(m) = \phi(1)^m$ and the order of $\phi(1)$ (in G) divides n.
   (b) List all homomorphisms $\mathbb{Z}_5 \to \mathbb{Z}_4$. (No proof necessary.)
   (c) List all homomorphisms $\mathbb{Z}_{10} \to \mathbb{Z}_6$. (No proof necessary.)
   (d) List all homomorphisms $\mathbb{Z}_{12} \to \mathbb{Z}_6$. (No proof necessary.)

5. Section 5 # 24, 39

6. Section 6 # 14, 18, 28, 32

7. (Challenge problem – optional!) Let G and G' be groups.
   (a) Prove or disprove: *if $\phi : G \to G'$ is a homomorphism, then for all $g \in G$, $\phi(g)^{-1} = \phi(g^{-1})$.*
   (b) Prove or disprove: *if $\phi : G \to G'$ is a function such that for all $g \in G$, $\phi(g)^{-1} = \phi(g^{-1})$, then ϕ is a homomorphism.*