Homework 10

Math 416 (sections 200 [Honors] and 500 [Regular]), Spring 2017

This homework is due on Thursday, March 30. *Write your section number on the top of your homework.*

0. *(This problem is not to be turned in.)*
   (a) Read Section 49–50
   (b) (Practice Problems) Section 49 #2
   (c) (Practice Problems) Section 50 #2, 7–9, 17, 22-23
1. True/False. (You do *not* need to give proofs for this problem.)
   (a) Let α and β be algebraic elements over a field F. Then $F(\alpha)$ and $F(\beta)$ are isomorphic fields if and only if their minimal polynomials over F are the same.
   (b) The splitting field of $x^4 - 3$ over $\mathbb{Q}$ is $\mathbb{Q}(\sqrt[4]{3})$.
   (c) The splitting field of $x^4 - 3$ over $\mathbb{R}$ is $\mathbb{C}$.
   (d) If E is a field extension of F, and both E and F are algebraically closed, then $E = F$.
   (e) Every algebraically closed field has characteristic 0.
   (f) $\mathbb{C}(x)$ is an algebraically closed field.
2. Section 49 #6, 8(not i), 11, 13
3. Section 50 #14
4. (a) Assume that E is a simple algebraic extension of a field F. What do $|G(E/F)|$ and $\{E : F\}$ count? Conclude that $|G(E/F)| \leq \{E : F\} \leq [E : F]$.
   (b) Compute $[\mathbb{Q}(\pi) : \mathbb{Q}(\pi^2)]$ and $\{\mathbb{Q}(\pi) : \mathbb{Q}(\pi^2)\}$. Explain your answers.
5. Compute the degree over $\mathbb{Q}$ of the splitting field of $x^3 - 1$ over $\mathbb{Q}$. Explain your answer.
6. Assume that E is a finite extension of a field F that is contained in $\overline{F}$. Prove that if $\{E : F\} = |G(E/F)|$, then E is a splitting field over F. (This is a partial converse of Corollary 50.7.)
7. Let E be the splitting field of $f(x) = x^3 - 3x + 1$ over $\mathbb{Q}$. Determine the group $G(E/\mathbb{Q})$. (*Hint*: Show that if α is a zero of f, then so is $\alpha^2 - 2$.)
8. Prove that the splitting field of $x^p - 1$ (where p is a prime) over $\mathbb{Q}$ is an extension of $\mathbb{Q}$ of degree $p - 1$. (*Hint*: Section 23.)

9. (Honors only!) Find the splitting field of $x^p - 2$ (where p is a prime) over $\mathbb{Q}$, and prove that it is an extension of $\mathbb{Q}$ of degree $p(p-1)$.

10. (Honors only!) Section 50 #16 (Note: there is a typo in the book; '$G(E/F)$' should be '$G(K/F)$'.)

11. (Honors only!) *Prove or disprove:* if E is a splitting field of a field F, then there exists a *unique* isomorphism of E that extends the identity map on F.

12. (Honors only!) Let E/F be an extension. Prove that if $[E : F] = 2$, then E is a splitting field over F.