Homework 13

Math 416 (sections 200 [Honors] and 500 [Regular]), Spring 2017

This homework is due on Thursday, April 20. *Write your section number on the top of your homework.*

0. *(This problem is not to be turned in.)*
   (a) Read Section 53
   (b) (Practice Problems) Section 53 #9, 11
   (c) Complete the following sentence, and give a proof:
   For $\alpha, \beta \in \overline{\mathbb{Q}}$, there exists an automorphism of $\overline{\mathbb{Q}}$ that fixes α but does *not* fix β if and only if __________________________.
   (d) Prove that any one-to-one homomorphism $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}$ is an automorphism.
   (e) *Prove or disprove*: $\overline{\mathbb{Q}} = \mathbb{C}$.
   (f) Are $\pm\sqrt[3]{2}$ conjugates over $\mathbb{Q}$?

1. Section 53 #1–8, 15, 16

2. Give an example of a field extension $F \subset E$ such that the group $G(E/F)$ is *not* abelian.

3. Assume that K is a finite normal extension of a field F of degree $p^n m$, where p is a prime number. Prove that there exist fields $F \subset E_1 \subset E_2 \subset \cdots \subset E_n \subset K$ with $[K : E_i] = p^{n-i+1}$ for $i = 1, 2, \ldots, n$.

4. Assume that K is a finite normal extension of a field F with Galois group $G := G(K/F)$. Assume that $|G| \neq 1$ and $|G| \neq p$ for any prime number p. Also, assume that for every subfield E with $F \subsetneq E \subsetneq K$, E is *not* a normal extension of F. Prove that for all prime numbers p that divide $|G|$, the group G has more than one Sylow p-subgroup.

5. (a) Write down a normal degree-3 extension of $\mathbb{Q}$. Explain. (Hint: a previous homework.)
   (b) Write down a degree-3 extension of $\mathbb{Q}$ that is *not* a normal extension. Explain.

6. Let K/F be a Galois extension with $G(K/F) \cong A_4$. Prove that there is *no* intermediate field $F \subseteq E \subseteq K$ with $[E : F] = 2$.

7. (Honors only!) Consider a degree-2 extension of a field that is not of characteristic 2. Prove that this is a normal extension.

8. (Honors only!) Assume that K is a finite normal extension of a field F with Galois group $G := G(K/F)$. Assume that K has *exactly* two distinct subfields E_1 and E_2 such that $F \subsetneq E_1, E_2 \subsetneq K$, and assume that E_1 and E_2 are both normal extensions of F. Prove that G is a cyclic group. (Hint: Section 37.)

9. (Honors only!)
   (a) Let K be a field, and let $f \in K[x]$ be a separable polynomial of degree n. Let L be the splitting field of f over K. Show that $\mathrm{Gal}(L/K)$ is isomorphic to a subgroup of S_n, and deduce that $[L : K] | n!$.
   (b) Let $K = \mathbb{Q}$ and $n = 3$ in part (a). Prove that if f is irreducible in $\mathbb{Q}[x]$ and has exactly one real root, then $\mathrm{Gal}(L/\mathbb{Q}) \cong S_3$.