Homework 2

Math 416 (sections 200 [Honors] and 500 [Regular]), Spring 2017

This homework is due on Thursday, January 26.

0. *(This problem is not to be turned in.)*
   (a) Read Sections 26 and 27.
   (b) (Practice Problems) Section 26 #4, 15, 30, 32
   (c) (Practice Problems) Section 27 #6, 24, 28
1. Section 26 # 10
2. Section 27 # 14
3. List all ring homomorphisms $\mathbb{Z}_{10} \to \mathbb{Z}_8$. Give a proof.
4. List all ideals of $\mathbb{C}$. Give a proof.
5. Is $\{(a,b) \mid a+b=0\}$ an ideal of $\mathbb{Z}^2$? Explain.
6. Prove or disprove the following:
   Claim: Let R be a ring. If there exists a nontrivial ring homomorphism $\phi : \mathbb{C} \to R$, then R contains a subring isomorphic to $\mathbb{C}$. (*Hint*: #4.)
7. (*Note*: The aim of this problem is to fill in the missing details at the end of the proof of Theorem 27.9.) Let I be an ideal of a ring R, and let $\gamma : R \to R/I$ be the usual projection homomorphism. (*Hint*: You may use #22 of Section 26 without proof.)
   (a) Use γ to define a function
   $$f : \{\text{ ideals of } R \text{ containing } I \} \to \{ \text{ ideals of } R/I \}\ .$$
   (b) Prove that f is a bijection.
   (c) Draw the following two conclusions:
      i. If N is an ideal of R that contains I such that $\gamma[N] = \{I\}$, then $N = I$.
      ii. If N is an ideal of R that contains I such that $\gamma[N] = R/I$, then $N = R$.
8. Let I be an ideal of a ring R. Prove that R/I is a commutative ring if and only if $ab - ba \in I$ for all $a, b \in R$.
9. Use the fundamental homomorphism theorem to prove that
   $$\mathbb{Z}_{20}/\langle 4\rangle \quad \cong \quad \mathbb{Z}_4\ .$$
10. (Honors only!) Prove or disprove the following:
    Claim: Let $\phi : R \to R'$ be a ring homomorphism. If R has unity 1, then $\phi(1)$ is the unity of R'.
11. (Honors only!) Prove or disprove the following:
    Claim: Let N and N' be ideals of a ring R. If $R/N \cong R/N'$, then $N = N'$.