Homework 3

Math 416 (sections 200 [Honors] and 500 [Regular]), Spring 2017

This homework is due on Thursday, February 2. *As a reminder, please write your section number on the top of your homework.*

0. *(This problem is not to be turned in.)*
   (a) (Practice Problems) Section 26 #29, 34
   (b) (Practice Problems) Section 27 #26
1. Section 27 # 30
2. Complete the following sentences:
   (a) The ideal $n\mathbb{Z}$ is a **prime** ideal of $\mathbb{Z}$ if and only if ____________.
   (b) The ideal $n\mathbb{Z}$ is a **maximal** ideal of $\mathbb{Z}$ if and only if ____________.
3. (a) Is the ideal $\langle 2x,\ 3\rangle$ a principal ideal of $\mathbb{Q}[x]$? Prove your answer.
   (b) Is the ideal $\langle 2x,\ 3\rangle$ a principal ideal of $\mathbb{Z}[x]$? Prove your answer.
4. Consider $\mathbb{R}[x]$, the ring of polynomials with real coefficients. Let
   $$N \quad := \quad \{f \in \mathbb{R}[x] \mid f(5) = f(7) = 0\}\ .$$
   Is N a prime ideal of $\mathbb{R}[x]$? Give a proof.
5. (a) Determine whether the following ring is a field, and give a proof:
   $$\mathbb{Q}[x,y]/\langle y-1,\ x+y+2\rangle.$$
   (Recall that $\mathbb{Q}[x,y]$ is the ring of polynomials in two variables, x, and y; one such polynomial is $f(x) = x^3y - 1/3$.)
   (b) Is the ideal $\langle y-1,\ x+y+2\rangle$ a maximal ideal of $\mathbb{Q}[x,y]$? Is it a prime ideal? Explain your answers.
6. Let P be a prime ideal in a ring R, and assume that P contains the intersection of two ideals I and J. Prove that P contains I or P contains J.
7. (a) Prove or disprove the following:
   Claim: Let I be an ideal of a ring R. If every ideal of R that contains I is a principal ideal, then every ideal of R/I is a principal ideal. (*Hint:* Use #7 from Homework 2.)
   (b) (Honors only!) Prove or disprove the following:
   Claim: Let I be an ideal of a ring R. If every ideal of R/I is a principal ideal, then every ideal of R that contains I is a principal ideal.

8. (Honors only!) Let R be a commutative ring with unity $1 \neq 0$. Prove that if $R[x]$ is an integral domain in which every ideal is a principal ideal, then R is a field.

9. (Honors only!) We know from class that every ideal in $\mathbb{Q}[x]$ is a principal ideal. Show directly (by finding a generator and proving your answer) that the following ideals are principal: $I = \langle x^2 + 1,\ x - 6\rangle$ and $J = \langle x^3 + 3x,\ 5x^2 + 15\rangle$.