Homework 5

Math 416 (sections 200 [Honors] and 500 [Regular]), Spring 2017

This homework is due on Thursday, February 16. *As a reminder, please write your section number on the top of your homework.*

0. *(This problem is not to be turned in.)*
   (a) Read Sections 31 and 33
   (b) (Practice Problems) Section 31 #8, 10, 22, 26
   (c) (Practice Problems) Section 33 #9, 11, 12, 13
   (d) Compute the irreducible polynomial of $\frac{1}{11+\sqrt{3}}$ over the field $\mathbb{Q}(\sqrt{3})$.

1. (a) Are $\mathbb{Q}(1+\sqrt{5})$ and $\mathbb{Q}(\sqrt{5})$ isomorphic fields? Explain your answer.
   (b) Are $\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(\sqrt{5})$ isomorphic fields? Explain your answer.

2. (a) How many $(3^3-1)^{\text{st}}$ (i.e., 26th) roots of unity does $GF(3^3)$ contain?
   (b) How many **primitive** $(3^3-1)^{\text{st}}$ (i.e., 26th) roots of unity does $GF(3^3)$ contain?
   (c) How many elements $\alpha \in GF(3^3)$ are there for which $GF(3^3) = \mathbb{Z}_3(\alpha)$?

3. Prove that for a field F, the following are equivalent:
   (i) F is algebraically closed,
   (ii) every nonconstant polynomial $f \in F[x]$ factors as a product of linear factors in $F[x]$, and
   (iii) every (nonconstant) irreducible polynomial is linear (degree 1).

4. For an element α in a field extension of a field F, prove that if $[F(\alpha) : F]$ is an odd number, then $F(\alpha) = F(\alpha^2)$. Give a proof.

5. Let E and K be two field extensions of a field F. Assume that $\alpha \in E$ and $\beta \in K$ are both algebraic over F. Prove that $\text{irr}(\alpha, F) = \text{irr}(\beta, F)$ if and only if there exists an isomorphism $\phi : F(\alpha) \to F(\beta)$ such that $\phi(\alpha) = \beta$ and $\phi|_F$ is the identity map of F.

6. Section 31 #19, 30

7. Section 33 #8, 10

8. (a) Is there a field of order 60? Explain your answer.
   (b) Compute $[GF(2^4) : \mathbb{Z}_2]$. Explain your answer.
   (c) Prove that if $GF(p^m) \subseteq GF(p^n)$, then m divides n.
   (d) (Honors only!) Prove the converse of part (c).

9. (Honors only!) Is $\mathbb{Q}(x^2, x^3)$ a simple extension of $\mathbb{Q}$? Explain.

10. (Honors only!) A finite subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of a field extension E of a field F is *algebraically independent over* F if there is no nonzero polynomial $f \in F[x_1, x_2, \ldots, x_n]$ for which $f(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0$.

    (a) Is $\{x^2, xy, y\} \subset \mathbb{Q}(x, y)$ algebraically independent over $\mathbb{Q}$? Explain.

    (b) Is $\{x^2, xy\} \subset \mathbb{Q}(x, y)$ algebraically independent over $\mathbb{Q}$? Explain.

    (c) Prove that a finite subset of E that is algebraically independent does *not* contain any element that is algebraic over F.

11. (Honors only!) Use Zorn's Lemma to prove:

    (i) Every ring with 1 contains a maximal ideal.

    (ii) Let W be a subspace of a vector space V. Then there exists a linear transformation $V \to V$ for which the kernel is W.