Homework 9

Math 416 (sections 200 [Honors] and 500 [Regular]), Spring 2017

This homework is due on Thursday, March 23. *Write your section number on the top of your homework.*

0. *(This problem is not to be turned in.)*
   (a) Read Section 48
   (b) (Practice Problems) Section 48 #7, 8, 23
   (c) Is there a field extension of $\mathbb{Z}_7$ that contains the imaginary number i?
   (d) Is $x^2 + 1$ irreducible in $\mathbb{Z}_5[x]$?
1. Is every group of order $455 = 5 \cdot 7 \cdot 13$ abelian? Prove your answer.
2. Is every group of order 160 *not* simple? Prove your answer.
3. Let G be a finite group. Let p be a prime number that divides the order of G. Assume that H is a p-subgroup of G and that K is a Sylow p-subgroup of G. Prove that H is isomorphic to a p-subgroup of K.
4. Section 48 #12, 20, 29, 35
5. (Honors only!) Section 48 #39
6. (Honors only!)
   (a) Prove the following: *For a finite group G and a prime p that divides $|G|$, if $|G|$ does not divide $(n_P)!$, then G is* **not** *simple.* (Hint: define a group action on the set of all Sylow p-subgroups of G.)
   (b) Use (a) to give a second proof for #2.
7. (Honors only!) The goal of this problem is to prove the Recognition Lemma we stated (but did not prove) in class: *If H and K are normal subgroups of a group G for which (1) $H \cap K = \{e\}$ and (2) $H \vee K = G$, then $G \cong H \times K$.*
   (a) Prove that for all $h \in H$ and $k \in K$, the following holds: $hk = kh$.
   (b) Prove that $\phi : H \times K \to G$, given by $(h, k) \mapsto hk$, is a group isomorphism.