Homework 12

Math 653, Fall 2019

This homework is due on Thursday, November 14.

1. Read Hungerford, Section 3.1.
   (a) What is a *group ring* (page 117)?
   (b) What does the *binomial theorem* (page 118) allow us to compute?
   (c) Section 3.1 #3, 6, 15
   (d) *(These problems are not to be turned in.)* Section 3.1 #1, 2, 11, 14, 18
2. The *center* of a ring R is $C(R) := \{z \in R \mid zr = rz \text{ for all } r \in R\}$.
   (a) Is the center of a ring always a *subring* of the ring? (Prove your answer.)
   (b) Is the center of a ring always an *ideal* of the ring? (Prove your answer.)
3. An element r in a ring is *nilpotent* if $r^n = 0$ for some positive integer n.
   (a) Prove that if R is a commutative ring and $r, s \in R$ are both nilpotent, then $r + s$ also is nilpotent.
   (b) Is (a) still true if R is non-commutative? Prove your answer.
   (c) Assume R is commutative. Does the set of nilpotent elements form an *ideal* of R? Prove your answer.
4. Let $\mathbb{F}$ be a field.
   (a) Let V be a vector space over $\mathbb{F}$. Let $\text{End}_{\mathbb{F}}(V)$ denote the set of linear transformations from V to V. Prove that $\text{End}_{\mathbb{F}}(V)$ is a ring under addition (of functions) and composition.
   (b) Assume that, additionally, V is finite-dimensional over $\mathbb{F}$; let n denote the dimension. Prove the following isomorphism of rings: $\text{End}_{\mathbb{F}}(V) \cong M_n(\mathbb{F})$.
5. Let $R = \mathbb{Z}_2[x]$ and I be the ideal of R generated by $x^2 + x + \overline{1}$.
   (a) Show how to identify R/I with the set $\{0,\ 1,\ x,\ x+1\}$. (Explain.)
   (b) Compute the addition and multiplitation tables for R/I. Is R/I a field? Explain.