Homework 2

Math 653, Fall 2019

This homework is due on Thursday, September 5.
You may cite results from class or previous homework, as appropriate.

1. Read the Hungerford, sections 1.1–1.4.
   (a) Section 1.1, # 7
   (b) Section 1.2, # 2, 5, 8
   (c) Section 1.3 # 1, 3
   (d) Section 1.4, # 4
2. *Prove or disprove*: The following **Borel group** is a subgroup of $\mathrm{GL}_n(\mathbb{R})$:
$$B_2(\mathbb{R}) \;:=\; \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{R}, ac \neq 0 \right\} .$$
3. Let $f : G \to H$ be a group homomorphism. Prove or disprove the following:
   (a) If f is surjective, then $|f(g)| = |g|$ for all $g \in G$.
   (b) If f is injective, then $|f(g)| = |g|$ for all $g \in G$.
4. Let $f : \mathbb{C}^* \to \mathrm{GL}_2(\mathbb{R})$ be defined by $f(a + bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ for all $a, b \in \mathbb{R}$.
   (a) Is f a group homomorphism? Prove your answer.
   (b) Is f injective? Surjective? Prove your answers.
   (c) Prove that $f^{-1}(\mathrm{SL}_2(\mathbb{R})) = S^1$, where S^1 denotes the unit circle in the complex plane.
5. Redefine a *group homomorphism* via a commutative diagram.
6. (a) Is $\mathbb{Z}$ a normal subgroup of $\mathbb{Q}$? Explain.
   (b) Is $\mathbb{Q}/\mathbb{Z}$ abelian? Infinite? Explain.
7. A subgroup H of a group G is *maximal* if $H \subsetneq G$ and there is no subgroup K of G such that $H \subsetneq K \subsetneq G$.
   (a) Which subgroups of $\mathbb{Z}$ are maximal? Explain.
   (b) Does $\mathbb{Q}$ have maximal subgroups? Prove your answer.
   (c) Are $\mathbb{Z}$ and $\mathbb{Q}$ isomorphic groups? Prove your answer.

8. Let H be a subgroup of a group G, and let $a, b \in G$. Consider the following claims: (1) $aH = bH$, (2) $a \in bH$, (3) $ab^{-1} \in H$, and (4) $ba^{-1} \in H$.

   (a) State all implications among the four claims.

   (b) Prove the implications in your answer to (a).

   (c) Prove that all remaining implications (if any) are false.

9. Let H be a subgroup of a group G, and let $g \in G$. Consider the following claims: (1) $gH = Hg$, (2) $gHg^{-1} = H$, (3) $gHg^{-1} \subseteq H$, and (4) $gHg^{-1} \supseteq H$.

   (a) State all implications among the four claims.

   (b) Prove the implications in your answer to (a).

   (c) Prove that all remaining implications (if any) are false.

   (d) How would your answers change if G is finite?

10. (a) Prove that a homomorphism $\phi : \langle a \rangle \to G$ from a cyclic group generated by a to a group G is uniquely determined by $\phi(a)$. (*Hint*: What is $\phi(a^n)$?)

    (b) List all homomorphisms $\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$. (No proof necessary.)

    (c) Let G be a group. Does the set of homomorphisms $\mathbb{Z} \to G$ have the same cardinality as G? Explain.

11. (a) Let G be a group. *Prove or disprove*: a function $\phi : \mathbb{Z}/n\mathbb{Z} \to G$ is a homomorphism if and only if $\phi(m) = \phi(1)^m$ and the order of $\phi(1)$ (in G) divides n.

    (b) List all homomorphisms $\mathbb{Z}/5\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$. (No proof necessary.)

    (c) List all homomorphisms $\mathbb{Z}/10\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$. (No proof necessary.)

    (d) List all homomorphisms $\mathbb{Z}/12\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$. (No proof necessary.)

12. Complete the group tables below.

Group Tables

	ρ_0	ρ_1	ρ_2	μ_1	μ_2	μ_3
identity= ρ_0						
ρ_1						
ρ_2						
μ_1						
μ_2						
μ_3						

Table 1: $S_3 \cong D_3$, where ρ_i's denote rotations and μ_i's denote flips

	ρ_0	ρ_1	ρ_2	ρ_3	μ_1	μ_2	δ_1	δ_2
identity= ρ_0								
ρ_1								
ρ_2								
ρ_3								
μ_1								
μ_2								
δ_1								
δ_2								

Table 2: D_4

	1	ρ	ρ^2	ρ^3	ρ^4	s	$s\rho$	$s\rho^2$	$s\rho^3$	$s\rho^4$
identity= 1										
ρ										
ρ^2										
ρ^3										
ρ^4										
s										
$s\rho$										
$s\rho^2$										
$s\rho^3$										
$s\rho^4$										

Table 3: $D_5 = \langle \rho, s \mid \rho^5 = s^2 = 1,\ \rho s = s\rho^{-1} \rangle$