Homework 3

Math 653, Fall 2019

This homework is due on Thursday, September 12.

1. Read Hungerford, section 1.5.
   (a) *Prove or disprove*: For subgroups H and K of a group G, the set HK is a subgroup of G if and only if $HK = KH$.
   (b) Prove Proposition 4.9 (on page 40).
   (c) Section 1.4 # 5, 8
   (d) Section 1.5 # 1, 16
2. Let p be a prime number. Use Lagrange's theorem to prove that (up to isomorphism) there is only one group of order p, and that in any such group every non-identity element generates the group.
3. Are $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$ isomorphic groups? Prove your answer.
4. *Prove or disprove*: Let H be a subgroup of a group G. Let $\mathcal{L}$ (respectively, $\mathcal{R}$) denote the set of left (respectively, right) cosets of H in G. Then the function $\mathcal{L} \to \mathcal{R}$ given by $gH \mapsto Hg$ is well defined.
5. Let G be a group, and let $g \in G$. Consider the function $\phi : G \to G$ given by $\phi(x) = gxg^{-1}$.
   (a) Prove that ϕ is a homomorphism.
   (b) Determine the kernel of ϕ.
   (c) Is ϕ an automorphism? Give a proof. (Recall that an *automorphism* of a group K is an isomorphism from K to K.)
6. Let G be a group. Define, for $g \in G$, the function $i_g : G \to G$ given by $i_g(x) := gxg^{-1}$. Let $I_G := \{i_g \mid g \in G\}$.
   (a) Prove that $\mathrm{Aut}(G)$, the set of all automorphisms of G, forms a group under composition.
   (b) Prove that I_G is a subgroup of $\mathrm{Aut}(G)$.
   (c) Prove that I_G is a *normal* subgroup of $\mathrm{Aut}(G)$.
7. List all automorphisms of $\mathbb{Z}_{12}$. No proof necessary.
8. Let H be a subgroup of a group G, and let $g \in G$.
   (a) Is gHg^{-1} always a subgroup of G? Prove your answer.

(b) Is gHg^{-1} always isomorphic to H? Prove your answer.

9. (a) Does the symmetric group S_7 have any cyclic subgroups of order 9? Explain.
   (b) What is the smallest n for which S_n contains a permutation of order 10? What about order 9? Explain.

10. Prove or disprove the following:
    (a) If $f : G \to H$ is a group homomorphism, and K is a subgroup of H, then $f^{-1}(K)$ is a subgroup of G.
    (b) If $f : G \to H$ is a group homomorphism, and K is a subgroup of H, then $f^{-1}(K)$ is a *normal* subgroup of G.
    (c) If $f : G \to H$ is a group homomorphism, and K is a *normal* subgroup of H, then $f^{-1}(K)$ is a *normal* subgroup of G.
    (d) If $f : G \to H$ is a group homomorphism, and L is a *normal* subgroup of G, then $f(L)$ is a *normal* subgroup of H.

11. Do the $n \times n$ elementary row-operation matrices of $GL(n, \mathbb{R})$ generate $GL(n, \mathbb{R})$? Explain.

12. Let G be the set of all functions $\mathbb{R} \to \mathbb{R}$.
    (a) Does composition make G into a group? Explain.
    (b) Does addition (of functions) make G into a group? Explain.
    (c) For each group structure/operation, is $H := \{f \in G \mid f(5) = 0\}$ a normal subgroup of G? (Explain.) If so, determine whether $G/H \cong \mathbb{R}$.

13. Let G be the group of all permutations of $\mathbb{Z}$. Let H be the subset of G containing all permutations that fix all nonpositive integers (i.e., $f(x) = x$ for all $x \leq 0$). Let $\sigma \in G$ be defined by $\sigma(x) := x + 1$ for all $x \in \mathbb{Z}$.
    (a) Is H a subgroup of G? Prove your answer.
    (b) Computer the left coset σH and the right coset $H\sigma$. Are they equal? Is one contained in the other?
    (c) How is this example related to Homework 2 # 9?

14. Read Ravi Vakil's advice on attending seminar talks: `http://math.stanford.edu/~vakil/potentialstudents.html`. What (if anything) surprised you? What do you hope to try when attending a future talk?