Homework 4

Math 653, Fall 2019

This homework is due on Thursday, September 19.

1. Read Hungerford, section 1.6.
   (a) Section 1.5, # 2, 5, 7
   (b) Section 1.6, # 1, 5, 6, 9
2. *Prove or disprove*: Every group having only finitely many subgroups is finite.
3. *Prove*: If N is a normal subgroup of G, then (1) the set of subgroups of G/N is $\{K/N \mid N \subseteq K < G\}$, and (2) K/N is normal in G/N if and only if K is normal in G. (*Hint*: Use a theorem from class.)
4. Let N be a normal subgroup of G. *Prove or disprove*:
   (a) If G is finitely generated, then so is G/N.
   (b) If N and G/N are finitely generated, then so is G.
5. Let H and K be subgroups of G. A *double coset* is $HxK := \{hxk \mid h \in H,\ k \in K\}$ (for some $x \in G$). Prove that any two double cosets are either disjoint or equal.
6. Let $\mathcal{F}$ be the set of (affine linear) functions $f_{a,b} : \mathbb{R} \to \mathbb{R}$ given by $f_{a,b}(x) := ax + b$, where $a, b \in \mathbb{R}$ with $a \neq 0$.
   (a) Which operation makes $\mathcal{F}$ into a group? Explain.
   (b) Prove that the set $\mathcal{T} := \{f_{a,b} \mid a = 1\}$ is a normal subgroup of $\mathcal{F}$.
   (c) What is $\mathcal{F}/\mathcal{T}$ (isomorphic to)? Prove your answer.
7. Compute $\mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2)$. (Recall the definition of automorphisms from Homework 3.)
8. Assume that H and K are normal subgroups of G. Prove that if G/H and G/K are abelian, then so is $G/(H \cap K)$.
9. Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, where $i^2 = -1$. Consider the following subgroup of $\mathrm{GL}_2(\mathbb{C})$, called the *quarternion group*: $Q_8 := \langle A, B \rangle$.
   (a) Prove that Q_8 is a nonabelian group. What is its order? Explain.
   (b) Is Q_8 isomorphic to $D_{2\cdot 4}$, the dihedral group of order 8?
10. *Prove or disprove*: $D_{2\cdot 12}$ (the dihedral group of order 24) is isomorphic to S_4.
11. (a) Give an example of a function $f : A \to B$ and a set $X \subset A$ where $f^{-1}(f(X)) \neq X$.
    (b) Give an example of a function $f : A \to B$ and a set $Y \subset B$ where $f(f^{-1}(Y)) \neq Y$.