Geometry and Topology: Seven Lectures by Raoul Bott

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ABSTRACT. These are notes for a series of lectures delivered by Professor Raoul Bott, William Caspar Graustein Professor of Mathematics at Harvard University, at City University of Hong Kong during the spring of 1999. The topics covered are: the role of shape in mathematics and physics, morse theory in mathematics and physics, and an introduction to equivariant cohomology. The first chapter of these notes, on the role of shape, is expository, while the second and third chapters push successively closer to the frontiers of current research in geometry and topology. The lecture for chapter one was a City University of Hong Kong Public Lecture delivered on March 4, 1999. The lecture for chapter two, presented on February 5, 1999, was sponsored by the Epson Foundation. The remaining lectures took place between February and April of 1999.

Contents

Foreword

For the benefit of the reader, let us begin with some brief biographical notes on Professor Raoul Bott. A much more detailed description can be found in Volume I of his collected works [**Bott1**].

Professor Bott was born on the 24th of September 1923 in Budapest. Following his childhood in the Hungarian part of Slovakia, he was sent by his stepparents to England in 1939. He then emigrated in 1940 to North America with his family just weeks before the bombing of England.

After receiving his doctoral degree from the Carnegie Institute of Technology in 1949, he was appointed to a research position at the Institute for Advanced Study at Princeton University. There, Professor Bott extended his interest in geometry by studying Morse theory from its inventor, Marston Morse. In 1951, Professor Bott moved to the University of Michigan, where he completed his research on the Periodicity Theorem, a major discovery opening up new directions in topology. In 1959, he was appointed Professor of Mathematics at Harvard University, eventually spending much of his career there. More recently, Professor Bott was appointed Higgins Professor of Mathematics in 1967 and William Caspar Graustein Professor of Mathematics in 1977.

Professor Bott has received numerous honors and awards, including the Oswald Veblen Prize in Mathematics, which was awarded to him by the American Mathematical Society in 1964 for his work on Periodicity. That same year, Professor Bott was elected to the National Academy of Sciences, and in 1987 he won the National Medal of Science for his career accomplishments. In 1990, the American Mathematical Society awarded Professor Bott the Steele Career Prize in further recognition of his outstanding work in geometry and topology. In 1995, he was elected to the French Academy of Sciences.

Of his many Ph.D. students, two have gone on to become Fields Medallists; namely, Professor Steve Smale (in 1966) and Professor Daniel Quillen (in 1978). Another former student of Professor Bott, Professor Robert MacPherson, was the 1992 recipient of the National Academy of Sciences Award in Mathematics.

The editor thanks Professor Steve Smale and our Dean, Professor Roderick Wong, for making Professor Bott's visit possible. Special thanks also go to Colette Lam for her great assistance in making **everything** (from videotaping, to organizing, to publishing, and more) go smoothly, and to Nancy Chan's help with the illustrations and the type-setting in chapter 3.

As a final personal note, I would like to add that it was a profound pleasure and honor to at last meet my mathematical grandfather. Professor Bott's warmth, as well as his mathematics, will long be remembered at City U of HK, and we all very much look forward to seeing him again.

Any errors or omissions are, of course, the sole responsibility of the editor.

J. Maurice Rojas
City University of Hong Kong
July 9, 2001

CHAPTER 1

The Notion of Shape in Mathematics and Physics

Abstract

A topologist studies those attributes of shape that are preserved under a large class of distortions. Such attributes are then called topological invariants of the shape, and in the simplest instances can be described numerically. I will try to explain the generation of some of these invariants, and the remarkable role they play in such disparate fields such as Modern Physics and the solution of equations over the integers.

The Public Lecture

It is not every day that a mathematician is asked to give a public lecture and I therefore consider it a great honour to address you in this forum to — so to speak — "put my case" to a general audience. I will try, then, to give you a glimpse into the concerns that fascinate some of us topologists and geometers, and on the way also hopefully give you some insight why a seemingly sane person might devote a life-time to a subject such as Abstract Mathematics.

Well, possibly the best place to start is with an anecdote from 35 years ago. In 1956 I was passing through San Francisco for the first time, a young penurious Postdoc on his first mathematical junket, and I was correspondingly lodgin in a very ramshackle old hotel. One day, as I made my way to my tiny room in the tiny elevator, I was joined by an older quite grim looking gentleman with walking stick. I asked him at which floor he would like to stop, and he said the 4th. Now I was going to the 4rd and so — possibly being over-polite — I asked him, "Is it alright to stop on the 3rd first?" He glared at me and somehow managing to shake his walking stick in front of my nose in those very cramped quarters, came out with:

> *"Three comes before four, even in this Crazy Mixed Up World!"*

My initial alarm quickly turned into delight! *Precisely,* I thought to myself – *and that is why I am a mathematician!* For there is no doubt that what motivates all of us in this profession is a certain wonderful rationality and permanence in our pursuit. And although the precise degree of solidity in our foundation will vary with our individual natures, we all need to feel firm ground under our feet, and in this search we are willing — again each according to their need — to descend to the careful exploration of *very primitive* concepts. Concepts, that often seem completely obvious and devoid of interest to the commonsense layman.

You may take the concept I want to start with as a case in point. Consider then the following thought experiment. You are confronted with a basket of oranges on your left and a similar basket of apples on your right. Your problem is to decide whether there are an equal number of fruit in the two baskets.

You will no doubt say, well that is easy enough, just count the fruit in each basket and see if you get the same answer. True enough, but there is an even more primitive — and very instructive — solution to this problem. Namely: Pick an orange with your left hand and an apple with your right hand and place them side by side on the table. Next, do the same thing again. And again. In fact keep this procedure up until one of the baskets if empty. When that occurs and you find the other empty also — then and only then can you assert with confidence that the number of fruit in the two baskets was equal.

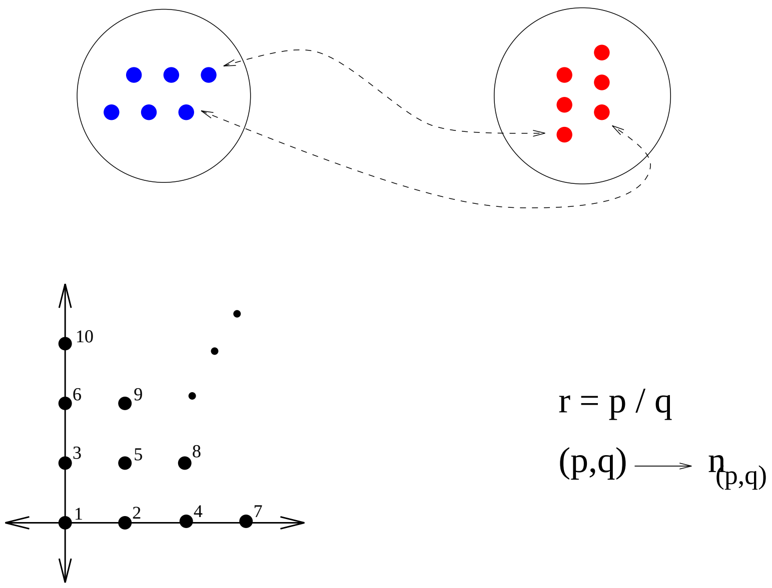

FIGURE 1. "Equal" in number $\Longleftrightarrow$ establishing a one-to-one correspondence. In particular, the rational numbers are equinumerous with the natural numbers.

In our mathematical jargon we say that by our fruit picking procedure you have *established a one-to-one correspondence* between the apples and the oranges in the two baskets. And in fact you should now see that quite generally "counting" amounts to no more or less than establishing a "one-to-one correspondence" between the *objects* you are counting and the *counting numbers* you hopefully carry permanently in your head. (That is — if you were educated in the dark ages before the advent of hand calculators.) Thus *one-to-one correspondence* is the basic concept underlying all everyday counting processes, and this concept of "equality via one-to-one correspondence" also carries over to all mathematical thought. Thus the logician will tell you that there are just as many even numbers: $2, 4, 6, \ldots$ as there are numbers: $1, 2, 3, \ldots$ Indeed the act of *doubling* clearly establishes a one-to-one correspondence: $1 \leftrightarrow 2$, $2 \leftrightarrow 4$, $3 \leftrightarrow 6$, etc... And we could easily spend the rest of this hour following this road into the realms of "set theory" and its many conundrums.

But as a topologist and geometer I am more concerned with "equality" in those domains, and to elucidate them, there is no better place than to start than with the perfect, Platonic world of Euclidean geometry. Indeed what do do we mean in this discipline when we declare triangles A and B *congruent*? It really means no more or less than that the points of A can be brought into *one-to-one correspondence* with those of B by a *translation followed by a rotation*. In more heuristic terms, "we can Platonically cut out the first triangle and then place it *precisely* over the second one."

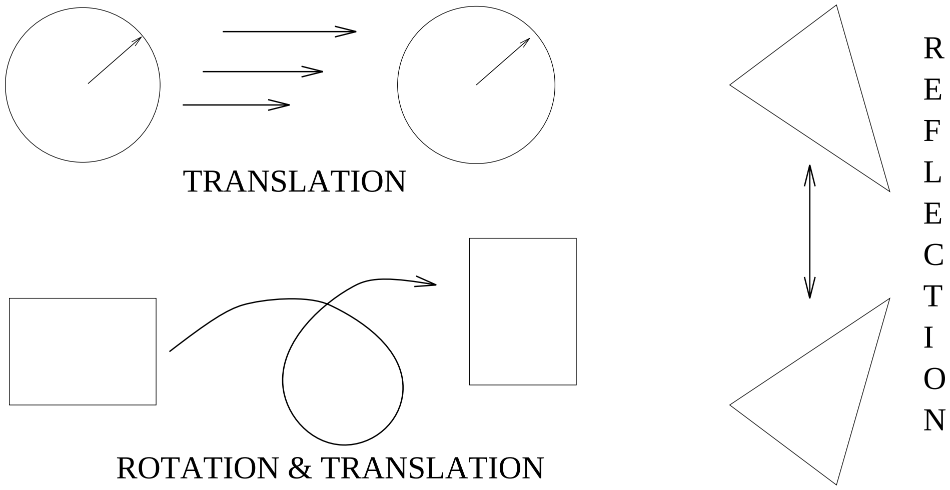

FIGURE 2. "Equal" in the Euclidean World $\Longleftrightarrow$ one-to-one via rotation and a translation.

It was Descarte who introduced numbers, and so eventually algebra, into the purely pictorial and Axiomatic Euclidean world. In more modern terms, he Digitalized Euclid, and like many inspired ideas, his idea is really quite simple. To explain it, let us start with the line. Once we have chosen a point p on it, and a unit of length, we can locate any other point q on the line by simply saying how many units q is away from p. We also agree that distance from p will be measured with postive numbers to the right and negative numbers to the left. This plan now realizes a

one-to-one correspondence of points on a line with the infinite decimals,

by associating to q the numerical distance it is from p in the above convention.

Similarly we can locate a point in the plane, by two numbers, after we have chosen a reference point and a set of perpendicular lines through that point. And similarly one needs three numbers to digitalize a point in this room, say. With this initial impetus, the Mathematician is naturally led to the concept of Euclidean n-space, $\mathbb{R}^n$, of *ordered n-tuples of real numbers*, where now a point is simply represented by n numbers:

$$q \leftrightarrow (x_1, \dots, x_n)\ ;\ n = 1, 2, 3, \dots$$

And it is fairly easy, at least in retrospect, to extend the ideas of Euclidean geometry, as we were taught them in high school, *by pure analogy*, into this n-dimensional domain. Thus we come to speak of regions in these domains as geometric objects.

For instance, the region in $\mathbb{R}^n$ of those n-tuples $(x_1, \dots, x_n)$ for which $x_1^2 + \cdots + x_n^2 = 1$ is called the $(n-1)$-sphere, because if we have only three numbers to deal with, this equation describes the surface of a ball.

Now topology is concerned with these geometric objects, but declares them to be "topologically equal" under a much more generous collection of one-to-one correspondences. Namely: *Two figures* (in the plane say) *are considered "topologically equivalent," or equal, if and only if their points can be put into one-to-one correspondence in such a way that "nearby points" in one figure go to "nearby points" of the other one.*

The rub here is of course the term "nearby," and it actually took the mathematical world several hundred years to figure out what precisely that should mean. But what this phrase is meant to capture, is that these *topological* one-to-one correspondences may *distort drastically*, but are not allowed to *tear*!

Here we will therefore have to do, for the most part, with some examples to illustrate these concepts. For instance, below you will find a list of topologically equivalent shapes in the plane.

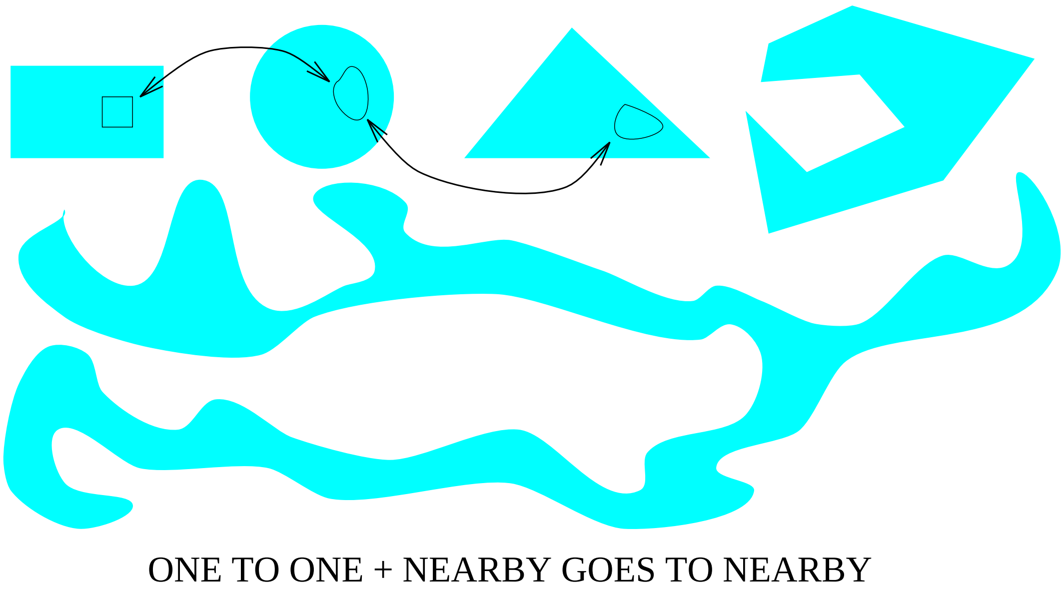

FIGURE 3. All these shapes in the plane are topologically equivalent.

On the other hand the following are **not** topologically equivalent:

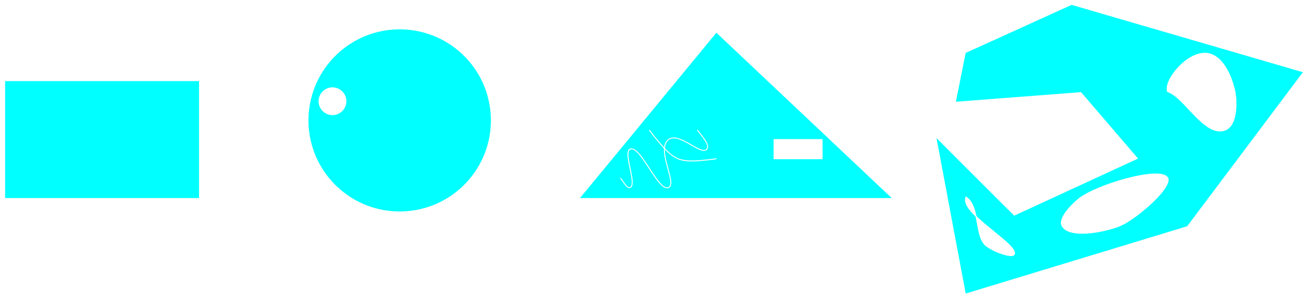

FIGURE 4. Any two of these four shapes in the plane are topologically **non**equivalent.

Notice then that the topologist does not "see" corners, and for him/her a straight line can be very crooked indeed by ordinary Euclidean standards.

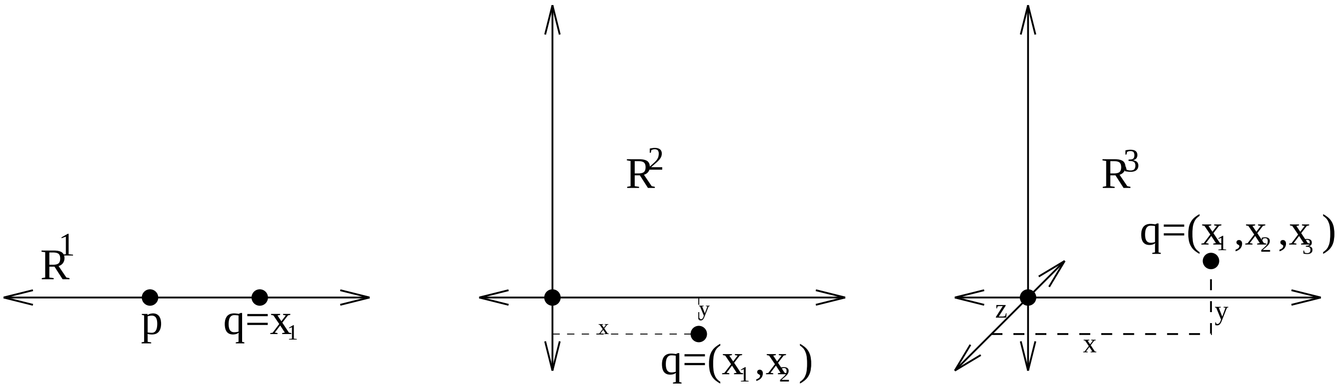

FIGURE 5. How Descartes began coordinatizing all Euclidean spaces...

After these examples let me actually bite the bullet and give you the definition of "nearby goes to nearby," just for the record, for I do not expect you to really digest it immediately. More of my calculus students usually can not do so even after a year! But here goes. A one-to-one correspondence of the points in the figure A with those of figure B takes "nearby points" to "nearby points" if and only if the following condition is fulfilled: Let p be a point in A and q be the point in B corresponding to it. Then for every circular disc D_1 in A of radius $r_1 > 0$ centered at p, we can find a second disk D_2 in B of radius $r_2 > 0$ centered at q so that under our correspondence all the the points of D_2 go over into points of D_1. Furthermore, the same propositions must be valid if A and B are interchanged.

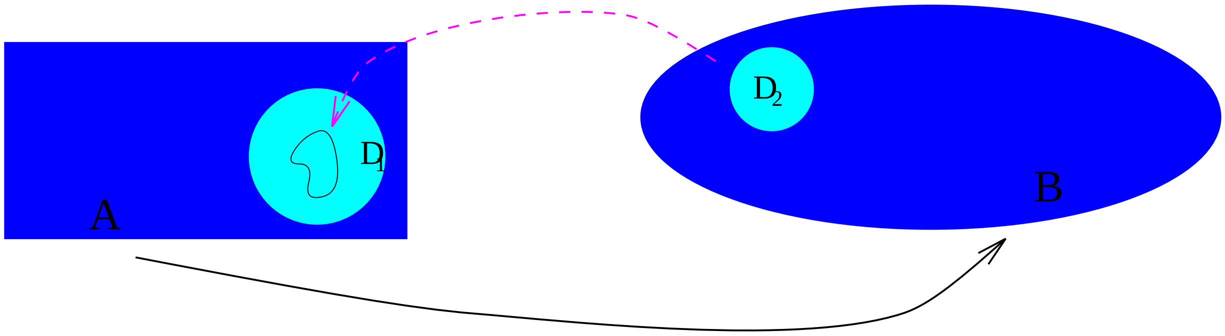

FIGURE 6. The definition of topological equality.

As I say, after years of thought, this concept has turned out to be the fruitful one and once laid down, allows for a precise and consistent development of the whole subject of *shape in Topology.*

On the other hand the Geometer and the Physicist, have found a rather more subtle notion of equivalence between geometric figures indispensable. To explain it consider the notion of *tangency* again in Euclidean geometry, to start with. Thus in the two figures below

We speak of a tangent line at p, as the line amongst all lines at p which best fits the circle at p. Similarly at q in the other figure, we have drawn a circle that *best fits* the ellipse at q. In fact Newton taught us how we can meaningfully define the notion that two curves are *tangent* to degree 1, 2, 3, etc... at a point.

Essentially this notion describes that as we zoom in on p, with higher and higher magnification, two curves that are tangent to the second order, say, look notably more alike than curves which agree only to the first order, etc... In any case — the *geometric* notion of equivalence is that of a *topological one-to-one correspondence which however preserves all orders of tangency for all curves.* Intuitively you should think of distortions which occur on a figure drawn on a rubber sheet, after you have stretched the rubber in various directions.

Thus the geometer can tell the difference between a square region and a "circular" region, but he still does not distinguish between the two "circular" regions indicated in the figure below:

Now, your burning questions are no doubt, why? Why all this fuss? What are these flights of thought good for?

Well, the Geometer's or Classical Physicists notion of equivalence is forced on us by an honest examination of the problem of describing the real world around us quantitatively, and in fact you are all quite used to examples of "geometric equivalence" from your everyday experience. For instance, consider any atlas. There you will find a series of figures representing different regions of the earth. Many countries will occur on various pages — but often in *different sizes* and quite possibly more seriously distorted. In short they are distorted, one against the other, but you will find that *all these distortions are indeed of the geometric kind.*

You might well say, that these are minor distortion compared to the more drastic ones we seem to be allowing. But *once you leave the rigid domain of Euclidean thought*, there is no really good way of speaking of "Mild Distortions." Geometric equivalence turns out to be the natural next step.

The Mathematicians have taken this step of studying objects up to geometric equivalence ever since Riemann, Gauss, Abel and Jacobi about 150 years ago, but these concepts underlie the very roots of the Physics of this century.

For the fundamental dictum of Einstein was precisely that the laws of Physics must be *"Distortion Proof!"* That is, *they must correspond under geometric equivalence.* For his view of our predicament was indeed very humbling! It asserts that however cunningly we might chart the phenomena in our vicinity, in space and time, we will only get a picture of these events — *up to geometric equivalence*! That is, a neighbouring observer will produce a chart of these same events, that will in general differ from ours by a *geometric distortion.*

Thus the Einsteinian world view starts with the assertion that by being careful we can make a *one-to-one assignment of the events near us in space and time — with four numbers*: the first three tell us the "where," relative to some measuring stick, and the fourth the "when" relative to some clock — in such a way, that the four numbers chart that any other observer makes of these same events, will differ by a *geometric equivalence.* In short, the universe is a *locally Euclidean space of dimension* 4, in the sense of geometric equivalence.

The positive side of this assertion is of course that it is not just a *topological equivalence*! So there is some basic faith expressed in this assertion and explicitly gives mathematical expression to the concept of "inertia" in Physics.

But of course the most radical "new" and revolutionary aspect of Einsteinian thought was that we can not disentangle time from space in the distortion of one observer relative to another! In short: *"God does not keep time for the universe,"* or as a famous Physicist friend of mine likes to put it to his class: *"In general relativity we do not know where we are, or what time it is."* It is only locally, in our vicinity, that we can disentangle space from time and it is only in these regimes and at moderate speeds, that Newton's laws, with the privileged role time plays in them, are valid. In some sense it is the same old story, of *Common Sense*, being drastically wrong! Just as common sense tells us that we live on a flat surface — but we actually live on the surface of a ball — so the self-centered view of the universe as being made up of *space and time* had to be discarded for the much more open-minded concept of a shape which is "locally given by four numbers," and whose local charts allow any kind of "geometric" distortion.

Now, quite on their own, already many years ago mathematicians puzzled about the question: What is the possible *global* shape of an object which is *locally* in geometric — or for that matter, topological — one-to-one correspondence with n-*tuples* of numbers?

To warm up let us consider the case $n = 2$, and I want you now to think about the following thought experiment. Suppose you are given an atlas in which each page describes four square miles of the surface of the earth. How could you tell that we live on the surface of a ball. In fact the following questions now arise naturaly: Can we make a list, a sort of zoo, of all locally "2-dimensional" objects, under geometric equivalence? And then under topological equivalence? Are the lists the same? And once we have this list, can we derive a procedure which, given an atlas for one of these objects, allows us to determine the global shape of that object?

Well the facts are these:

In dimension 1, that is $n = 1$, there is only one basic, *closed* topological or geometric shape. That is, the circle:

For $n = 2$ the basic shapes, again either under topological or geometric equivalence fall into two families: the first consists of the sphere and the infinite sequence of **k-holed tori** (a.k.a. **k-handled 2-spheres**) for $k \in \{1, 2, 3, \dots\}$. These are drawn (somewhat roughly) in figure 7 below. The second family consists of quotients of the first family, in the sense that every surface from the second family admits a continuous two to one map (covering every point) from a unique surface from the first family. The first family is **orientable**, in the sense that each of its surface has a rigorously definable "inside" and "outside", while all the surfaces of the second family lack this property. In particular, non-orientable surfaces admit simple pictorial illustrations only if one is allowed to make the surface "pass through itself". For example, consider **Boy's surface**, which is a representation of the non-orientable surface built as a quotient of the 2-sphere (see figure 8 below). However, the corresponding non-orientable surface, known as the **real projective plane**, **can** be embedded in four-dimensional Euclidean space.

In any event, as a consequence of the preceding classification, any competent topologist could, after being presented with the atlas under consideration, reassure you that you had

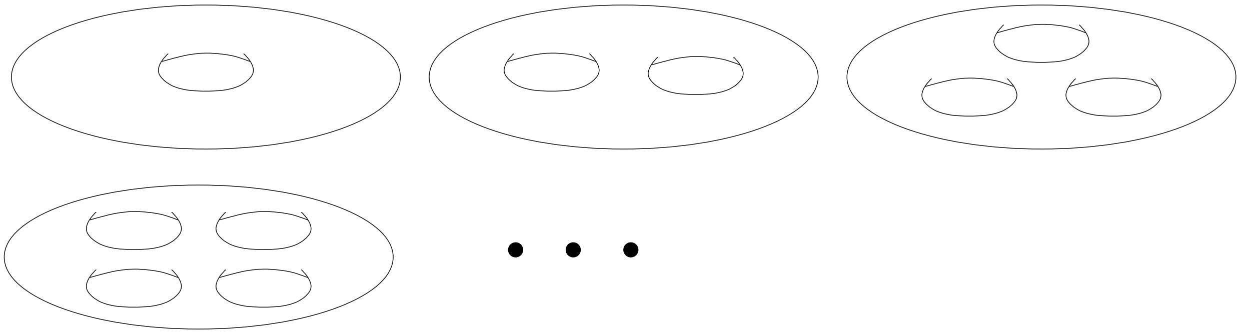

FIGURE 7. The compact orientable 2-manifolds, sans the 2-sphere.

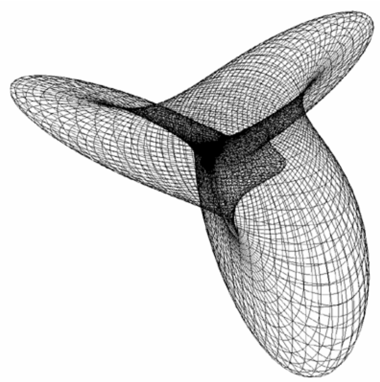

FIGURE 8. The real projective plane can **not** be embedded in $\mathbb{R}^3$, unless one allows a self-intersecting surface. (This mesh representation was generated by Paul Bourke, from the Swinburne Australia Astrophysics and Supercomputing Centre, in October 1992.)

an atlas for one of the shapes in the *first* family. However, to go further and tell you which one, some global procedure is needed.

But, he/she can also give you an algorithm for deciding this "global shape!" And in fact this recipe goes back to Descartes and Euler! Their principle is as follows: Take any one of the shapes from our first family of two-dimensional shapes, say the first one, the so called 2-sphere, that is, the surface of a ball. Divide this surface into triangular regions — i.e., into countries — and then make the following count:

$$\chi(S^2) = a_0 - a_1 + a_2,$$

where a_0 denotes the number of vertices, a_1 the number of edges, and a_2 the number regions of countries in your dissection. *I claim the answer will be* 2*, however you divide up the shape into triangular regions!*

For example let us represent the surface of the earth by one atlas containing only two maps, one representing the Northern hemisphere, and one the Southern hemisphere: and let us "triangulate" them as indicated below

Then clearly

$$\begin{aligned} a_0 &= 5+2=7 \\ a_1 &= 5+5+5=15 \\ a_2 &= 10 \end{aligned}$$

so that $\chi(S^2) = 7 - 15 + 10 = 2$.

I will leave you to experiment with other "triangulations" at home. Take an old tennis ball and pen, draw triangular countries on it and I give you a money back guarantee that after you make the proper count the answer for $\chi(S^2)$ will be 2.

On the other hand if you do the same for the surface of an inflated inner tube, then you will find that this magical number χ — called the Euler Characteristic — will turn out to be 0. And in general, for the nth surface on the upper list χ turns out to be $4-2n$.

Now I think it is clear that this fact conceptually solves the problem of deciding "which atlas describes which shape." We "triangulate" the region described on the first page. Then we look on the second page and mark the triangulation that we can see there corresponding to parts of the second map which already occurred in the first page. Once we have marked these, we continue to triangulate the rest of the second page. Then onto the next, and the next page till we are done. Finally we count all the vertices and edges and triangles, but with care not to count any of the objects "multiply."

I mentioned before that these basic two-dimensional shapes were already "known" in one way or another to such giants as Abel, Gauss, Riemann 150 years ago, but for a truly solid foundation we had to wait until this century, and it was only after this foundation was laid that the general search for a classification of higher dimensional shapes really began. And hand in hand with this search, goes the quest to find the algorithms needed to distinguish these "shapes" be it under "geometric" or "topological" equivalence.

These then have, by and large, been the main questions that have occupied the topologists and geometers of this century. Indeed they have kept us very busy, and we have also been much encouraged by seeing that the techniques we have developed along the way, have had a decisive influence not only on sister branches of mathematics, such as Algebra and Number Theory, but also on much of the world view of the present day Physicist.

On the other hand we are very far from our goal! Already in dimension three, we have not succeeded in finding a sufficient number of topological algorithms, to distinguish the simplest closed shape, the three-sphere, S^3, that is, the analogue of the boundary of a four-dimensional ball, from the other closed shapes! In fact one of the most famous of topological conjectures, the "Poincare Conjecture" to the effect that amongst the closed three-dimensional shapes, S^3 is characterised by the fact that any "closed curve" on S^3 can be shrunk to a point on S^3, has alluded us. And I might add: with a vengance. For all too many gifted topologists have become grey in its fruitless persuit.

However, along the way we have also come upon many very exciting surprises. One of these certainly was that in dimension *greater than four the Poincare Conjecture is true.* In other words, certain phenomenae become *easier* in higher dimensions! Another great surprise was that the topological and geometric classifications, which agree in dimensions 1, 2, and 3, fail in dimension 4 and higher! Furthermore they fail even for our simplest "closed" examples, the spheres, in dimension 7.

All these advances have come in the last 50 years and I well remember the excitement we all experienced when they were announced. The higher Poincare Conjecture, by Steven Smale; the different geometric structures on the topological 7-Sphere by John Milnor; and the different geometric structures on Euclidean 4-space as a consequence of the work of Michael Freedman and Simon Donaldson. Each one of these was rewarded with a well-deserved Fields Medal by the international community.

Now all these achievements in higher dimensions might strike you as being of the "Mount Everest" kind, the very question no more than an irresistible temptation for romantic young men and women to show their mettle. And in a court of law, I might have to fall back upon such a justification. But the truly amazing phenomenon in mathematics is precisely that many of these flights of "pure thought," initially motivated by purely aesthetic considerations have, over and over again, turned out to be useful for the more mundane — or should I say, the really *important* — aspects of our culture.

Let me give you my own experience as an example. I started out as an electrical engineer, because of my early fascination with electricity, which made me an expert at blowing fuses — and sometimes repairing them — by the time I was eight. After my degree from McGill and a short stint in the army during the 2nd world war, I returned to graduate school and there was captivated by the equations of "Maxwell" which so wonderfully present us with a mathematical description of the subtle behaviour of electricity — at roughly the scale of our own size. These equations also turned out to be invariant under the distortions of "special relativity," and in a sense these equation seduced me into wanting to become an applied mathematican or physicist or maybe astronomer. But when a few years later, after getting my doctorate in applied mathematics from Carnegie Tech, I found my self miraculously at the Institute of Advanced Study in Princeton — at a very interesting time in fact, with Einstein still vigorous and Oppenheimer the new energetic director — I *perversely* fell under the spell of Topology and Geometry, a subject that has continued to fascinate me ever since. Still, I find that now, forty years later, it is precisely these topological and geometric concepts that are indispensable for understanding of contemporary attempts to present a reasonably coherent mathematical picture of the world. A picture that is compatible with the "very small" and the "very large."

In fact the whole development of Mathematical Physics in the last 40 years has come to rely more and more on geometry and topology, except that these brilliant Physicists go us one better. Their insights are often generalizations of our insights in *finite* dimensions, to *"infinite"* dimensions!

All this started from the extraordinary phenomenae that were discovered at the start of this century, and which blossomed into Quantum Mechanics in the 20's and 30's. Whereas the experiment that was decisive for ushering in the theory of relativity established beyond any doubt that the behaviour of light contradicted Newtonian ideas — it turned out that the speed of light could not be speeded up or slowed down by moving the source — the Quantum Mechanical experiments defied "common sense" to a much, much greater extent. In particular it made the Platonic conception of a "point" unacceptable in the "real" world. Indeed these experiments showed that one could never, at the same time, determine the position and velocity of, say, an electron. The more precisely we measure one the more the measurement interferes with the value of the other. The attempt to bring this inherent "chaos in the small" into harmony with a seemingly deterministic universe "in the large" has led to the most bizarre philosophical proposals during the last 70 years, but is really still completely unresolved.

I can not attempt to describe the sort of mathematics that eventually led to the great triumphs of the Quantum Theory: Its picture of the atom, its explanation of the periodic table of the elements, or the mysterious relation between heat and light. But one would have expected that with the Platonic concept of a *"point"* so undermined, that such relatively naive concepts as the *"locally Euclidean shapes"* which we have been discussing, would have become quite obsolete. But that is not the case, at least at this stage of the game! If anything, Topology and Geometry entered Quantum Theory even more decisively than it had entered classical theory — *but through another door!*

Already in the 30's the very important attribute of "spin" of an eletron was found to be related to a definitely Topological notion, namely that of a *"covering space,"* and to give you a feeling for this concept, let me return to our table of closed locally Euclidean, 2 dimensional shapes.

You may recall that in this zoo there were two lines, and our discussion so far only dealt with the upper one. What about the lower line? Well, each surface in the lower line can be thought of as a "topological half" of the one above it. More precisely, there is a correspondence which to *each* point of the lower surface attaches *two* distinct points of the upper one! In our jargon we say that the upper one is a *double* cover of the lower one. Let me illustrate on our first surface, the 2-sphere, that is, the surface of a ball in Euclidean 3-space.

On this surface, every point p has an *antipode* $A(p)$, as indicated in the figure below:
and the surface, which we have denoted by $0/2$ in our table, is obtained from S^2 by *gluing* every p to its antipode $A(p)$! We may equally well describe this surface as the shape obtained from the upper hemisphere, but now only with opposite points on the *equator* glued together. See figure

The resulting space is locally Euclidean — indeed there is no problem during the gluing really — but it turns out that we can **not** completely carry out the gluing in 3-space! In short this is a *perfectly good* 2*-dimensional closed shape — but one which does not* "fit" *into* 3*-dimensional space!*

This shape is called the *projective plane* and it has many wonderful properties in geometry as well as in topology. In any case, all the "lower" surfaces in our zoo are obtained from the ones above by gluing appropriate "antipodes" together. These lower surface are also all examples of, so called, "unorientable shapes," and are all closely related to an example I am sure you have all seen at one time or another: That is, the *Möbius Strip*,
in which we glue opposite sides of a rectangular strip together, but only after giving the strip an odd number of twists.

As a little homework, I suggest that you convince yourself that, once a little disc is removed from the projective plane, the rest constitutes precisely a "Möbius strip." Hence the projective plane is a shape obtained from the Mobius strip by gluing the boundary of our ordinary disc to the boundary of the Möbius Strip! This is locally easy enough — but can not be done globally in 3-space.

But to return to the main point, in our Zoo the upper surfaces appear as "nontrivial" double covers of the lower ones. Nontrivial, because we could trivially create a one-to-**two** correspondence by simply taking *two copies* of the lower space. In any case, it is this concept of a nontrivial "double covering" that enters into the concept of Electron Spin!

More precisely, it is already a beautiful geometric insight, that the "totality of rotations of 3-space", constitute a *closed 3-dimensional locally Euclidean shape*, and is in fact in geometric one-to-one correspondence with the "projective space" obtained from the 3-sphere by the antipodal glueing procedure! Hence, the 3-sphere is a nontrivial double cover of the projective space and it is this phenomenon which in some sense "explains" the quantum mechanical spin of an electron. Not only that. In fact the example of the rotation group as a locally Euclidean shape, is decisive for all the mathematics of modern Nuclear Physics, in the sense that the Quark, the Gluon , etc., are each, in one way or another, associated to a definite generalization of the rotations of Euclidean 3-space qua "locally Euclidean closed shapes."

To explain why the rotation group corresponds to projective space, recall that any rotation always has an axis and is given by an angle of rotation about that axis. Hence we get a one-to-one correspondence of the possible rotations, by marking the magnitude of the angle of rotation along its axis:
But now the ambiguity of the angle of rotation — a 180 degree rotation is equal to a -180 degree rotation — forces us to identify the point 180 units "up" the axis with the one 180 units "down" the axis. In short to get a one-to-one correspondence, with all possible rotations, we must start with a *solid ball* and *identify antipodal points* on its boundary! This is clearly the analogue of what we did earlier in one dimension lower.

But notice that the projective space in this guise now has a quite new attribute, and one that its lower-dimensional analogue does not have! Namely if *we perform two rotations, one after the other, we again get a rotation*! So this shape admits some sort of a law of composition — a law we call the "*group law*." Objects which have this dual nature of being a locally Euclidean shape as well as having a group law, are called *Lie Groups* and their study has been a central concern of mathematicians of all stripes during this century. In fact one of the great achievements of the mathematics of this century is that we do have a complete Zoo of *all the closed locally Euclidean shapes which are Lie groups.*

You may think of these objects as the "perfect Solids" of our century, and it is an especially fascinating story how this purely mathematical classification has been incorporated into modern Quantum Theory.

Already in the twenties the attempt to bring the Maxwell equations into harmony with the Einsteinian law of gravity brought the circle — which is the simplest Lie Group — into the discussion, as being the hidden geometric object behind electricity. Its one dimensional structure here corresponds to the "invariance of charge" of an electron, under the famous principle of Emmy Noether, that continuous *symmetries* of dynamical equations correspond to "conserved quantities."

In the later developments, as new conserved quantities were found in the scattering experiments of cyclotrons new forces were postulated — corresponding to the more complicated

Lie groups. For instance the electro-weak force is, essentially, modelled by the Lie group corresponding to the 3-sphere S^3, and the strong Nuclear Force by an even more complicated Lie group called SV(3). But maybe, most amazing of all, the most exotic of our Lie Groups — the so called "Exceptional Lie group E_8" is intimately related to the "string theory" which has played such a fundamental role in the rethinking of the foundation of particle Physics, and has spawned such amazing propositions as that the world is really 10-dimensional with 6 "hidden"— or compactified — dimensions. Dimensions of which we are not aware, because of Quantum Mechanical effects.

But enough of Physics and its relations to topology and geometry. Let me close with a remark on the the connection of these disciplines to Number Theory, in a way the most concrete and at the same time most abstract of subjects. Concrete, because it asks questions about our counting numbers, 1, 2, etc., and abstract because the solution of these questions often involves all branches of mathematics. It is also the subject which has most steadfastly rejected any aspirations at being useful. Alas, perverse as their instinct was, they had not counted on the perversity, dishonesty and greed of humanity as a whole. In fact number theorists are now probably the most sought after of mathematicians in the *real* world. Their cunning algorithms are the only ones that might protect our privacy and the integrity of all financial transactions in the coming, computer driven, centuries.

You most probably have heard that in the last few years a very famous old problem in number theory has been solved. This is Fermat's assertion that the equation

$$x^n + y^n = z^n$$

has no solutions in the counting numbers 1, 2 ,3... as soon as n is greater than 2!

The irresistable allure to a mathematician of this question is of course that it deals with an infinite number of cases, and so, for instance, is quite out of range of the present day computer. Considerable allure was also added to this question by a remark of Fermat's in the margin of one of his books to the effect that he had just discovered a very simple proof of this assertion. Well no such simple proof has been found to this day, but Andrew Wiles discovered a very, very difficult, profound, and roundabout one a few years ago.

My comment here is simply that this theorem is also closly linked to our 2-dimensional shapes! This comes about in the following manner. If we allow x, y, and z to range over the usual number system, that is the infinite decimals, then the equation above has plenty of solutions, for all n! However, there is another miracle that was discovered by mathematicians over a hundred years ago. Namely, that all the properties we associate to number, as a point on the line, *extend* to *points* in the *plane*. Hence we can also look for solutions of our equation in this larger domain, and once this is done, every equation of the type written above, chisels out a definite shape in our Zoo, and another very beautifull theorem of the last 20 years, due to Gerd Faltings, asserts that if *the shape* that is *thus associated to an equation is of the higher kind than the first one, then it can only have a finite number of solutions in the realm of counting numbers.*

But now, I believe, it is really time to stop. Thank you for having been such a well-behaved audience. And my thanks are especially due to those, hopefully few, who have had — absolutely no idea — of what I have been talking about. To them I can only extend a hand of camaraderie. For the state of utter puzzlement is one we mathematicians know only too well. In fact many of us find ourselves in that state most of the time. However with us this state only serves as an incentive to get to the bottom of things and I hope that in some small way my remarks will have the same effect on you.

CHAPTER 2

Morse Theory in Mathematics and Physics

Abstract

Extrema of functions play a vital role in both topology and classical physics. Their role in the Quantum Theory is rather more subtle. This lecture attempts to contrast the various sides of this subject.

The Epson Foundation Lecture

Editor's Note: *This lecture was introduced by Professor S. Y. Cheng, President of the Hong Kong Mathematical Society and Full Professor at the Hong Kong University of Science and Technology.* $\diamond$

I started as an engineer and only came to mathematics later. Then, in 1949 at Princeton, I learned Morse Theory. To be more precise, I learned Morse Theory behind Morse's back...

Let us consider a manifold M and a point $p \in M$. Let $f : M \longrightarrow \mathbb{R}$ be a differentiable map and assume $\frac{\partial f}{\partial x}|_p = 0$. We call f **nondegenerate** iff $df_p = 0$, i.e., $\det[\frac{\partial^2 f}{\partial x^i \partial x^j}|_p] \neq 0$.

Morse had the good idea of counting critical points, and the preceding condition implies that there are only finitely many critical points on M when M is compact. In particular, consider

$$M_t(f) := \sum_{df|_p=0} t^{\lambda(p)} \ , \ \lambda_p := \# \text{ of negative eigenvalues of } [\frac{\partial^2 f}{\partial x^i \partial x^j}|_p]$$

One can even coordinatize locally so that $f(x) = f(p) + x_1^2 + \cdots + x_k^2 - y_1^2 - \cdots - y_\ell^2$, where $\ell := \lambda(p)$.

Examples: *Consider the two manifolds embedded in $\mathbb{R}^3$ illustrated in figure 1 below. If, for either manifold, we let f denote the "height" function $f(x_1, x_2, x_3) := x_3$, it is easy to see that the left and right manifolds respectively give us $M_t(f) = 2 + 5t + t^2$ and $M_t(f) = 1 + 4t + t^2$. We thus see that the function M_t is* **not** *a topological invariant. On the other hand, the change of coefficients is far from arbitrary: the* **Morse inequalities** *strongly restrict how much M_t can change once the topology of M is fixed.* $\diamond$

Let K be any field of characteristic zero, e.g., $\mathbb{Q}$. Let us then define $P_t(M) := \sum_i t^i \dim H^i(M; K)$.

Morse Inequality. *$M_t(f) - P_t(M; K) = (1 + t)Q(t)$, where $Q(t) = a_0 + a_1 t + \cdots$ and $a_i \geq 0$ for all i.* ■

An immediate corollary of this elegant inequality is the fact that $M_{-1}(f) = P_{-1}(M; K)$, which in turn is just the Euler characteristic of M. (This is easily verified for our examples above.) In particular, we also obtain that the Euler characteristic of M is independent of the field from which we pick the coefficients of our cohomology groups.

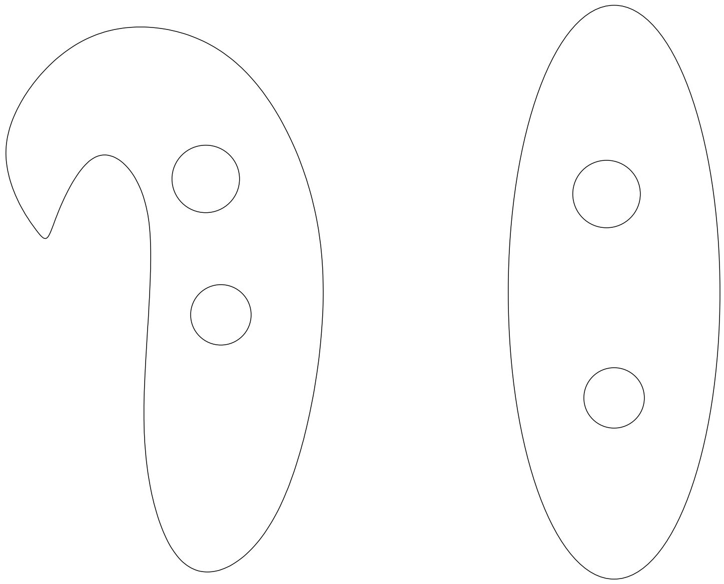

FIGURE 1. Consider $M_t(f)$ for each of these 2-tori embedded in $\mathbb{R}^3$, where f measures altitude...

Note also that if $M_t(f) = \sum_i m_i t^i$ with $m_i m_{i+1} = 0$ (i.e., the sequence of nonzero m_i is sparse in a suitable sense) then $Q(t) = 0$ identically. So we obtain that $M_t(f) = P_t(M; K)$ in this "lacunary" case. In short, it is sometimes enough to find a sufficiently good Morse function to do topology.

André Weil would have beamed at our next example.

EXAMPLE 1. *If $M = S^n$ (the standard unit n-sphere in $\mathbb{R}^{n+1}$ and f is the usual altitude function x_{n+1}, we easily obtain $M_t(f) = 1 + t^n$. So Morse theory is essentially the simplest way to compute the homology of S^n.* $\diamond$

EXAMPLE 2. *Take $M = \mathbb{P}^{n-1}_{\mathbb{R}}$ and $f(x) = \frac{\langle Ax,x\rangle}{\langle x,x\rangle}$, where $\langle\cdot,\cdot\rangle$ denotes the usual Euclidean inner product on $\mathbb{R}^n$ and A is a real $n \times n$ matrix. What are the extrema of f? Well a physicist would instead look at*

$$\tilde{f}(x) = \langle Ax, x\rangle - \lambda\langle x, x\rangle,$$

which in essence gives us an equivalent Langrange multipliers problem. It is then easily shown that the extrema occur precisely when λ is one of the eigenvalues of A. Assuming A is symmetric, it is well-known [**Str98**] *that A has exactly n distinct eigenvalues, all of which are real. So let us order them as $\lambda_0 < \cdots < \lambda_{n-1}$. So we easily obtain that $M_t(f) = 1+t+\cdots+t^{n-1}$, and thus $P_t(\mathbb{P}^{n-1}_{\mathbb{R}}; \mathbb{Z}/2\mathbb{Z}) = 1 + t + \cdots + t^{n-1}$, taking advantage of the fact that $2 = 0$ in the field of two elements.* $\diamond$

EXAMPLE 3. *Via almost the same reasoning as our last example, we can derive that $M_t(f) = 1 + t^2 + t^4 + \cdots + t^{2(n-1)}$, when $M = \mathbb{P}^{n-1}_{\mathbb{C}}$ and A is a complex* **Hermitian** *matrix. So in this case we have lacunarity and we can completely describe the cohomology when the coefficient field is $\mathbb{Z}/2\mathbb{Z}$.* $\diamond$

That was baby Morse theory...

So can one still compute cohomology in general? Yes, via the Morse-Smale-Milnor-Witten complex.

In this case, we pick a vector field in addition to a Morse-function. Assume $X_f \leq_t 0$ and that one understands the solutions of all the resulting differential equations (for integral curves). Physicists call these **instantons**. i.e., we have $\frac{\partial u}{\partial t} = -X_{\mu(t)}$ and the solutions are maps from $\mathbb{R}$ to a space of integral curves.

Most of the time, the solutions are nonexistent, alive for an instant, then dead most of the time. (Most of the time at one end point, then flashes over to the other, then most of the time at the new endpoint.)

If you know all the integral curves that go from a critical point of index p to a critical point of index $p+1$ (and you're in the generic case), then you can...

More precisely, let's construct a chain complex as follows: Set $C_q(f) = \bigoplus_{p\,:\,\lambda(p)=q} \mathbb{Z}$ and let the boundary operator be such that $\partial(1_p) = \sum_\mu \pm_q$ where the sum is over those μ satisfying $\mu(-\infty) = p$, $\mu(+\infty) = q$, and $\lambda(p) = \lambda(q) + 1$. We can then compute cohomology with coefficients in $\mathbb{Z}$.

Morse Theory (Classical Physics)

The most classical problem of physics is understanding the motion of particles acted upon by a force.

The **Lagrangian** is an object depending on a point and a direction on M. In this setting, the manifold M is usually taken to be some sort of configuration space.

Hamilton's Principle can be summarized by the following three eqations:

$$S(\mu) = \int_{t_0}^{t_1} L(\mu(t), \dot{\mu}(t))dt$$

$$\delta S = 0$$

where the first equation describes the energy necessary for our particle to move, from time t_0 to time t_1, along the path defined by μ. Note that the second equation is actually variational derivative depending on changes in the **path** μ — so motions of our particle correspond to critical points in the space of paths.

Morse understood this and discovered that his indices worked in this setting as well!

First, let $\Omega_{P,Q}$ denote the space of all (time 1) paths from P to Q, $S : \Omega_{P,Q} \longrightarrow \mathbb{R}$, and $M_t(f) = \sum_{dS=0} t^{\lambda(p)}$. $P_t(\Omega_{P,Q}M)$ is defined similarly...

(So now, a point-tangent pair is actually a **path with a tangent field**, but now the analogue of the Hessian is a particular differential operator: $H_f = \frac{\partial^2}{\partial x^i \partial x^j}$, which is somehow equivalent to $\frac{d^2 y}{dt^2} + q(y)Y = \lambda Y$, which becomes a classical Sturm problem.)

But nobody had computed the cohology of such infinite-dimensional loop spaces! Furthermore, changing f to $-f$ is not allowed any more, but now there are only finitely many eigenvalues...

Let's look at an example: S^n with $n > 2$. Let's look at the underlying geodesic problem... We must find the corresponding differential equations, count negative eigenvalues, etc... In the end, we obtain $M_t(f) = 1 + t^{n-1} + t^{2(n-1)} + \cdots = \frac{1}{1-t^n}$. So for $k > n$, lacunarity implies that $P_t(\Omega_{P,Q}) = \frac{1}{1-t^{n+1}}$. So on an n-sphere, with $n > 2$, there exist infinitely many geodesics between two points.

However, in the quantum theory (which is what we really want to consider) the Morse theory takes on a different flavor...

L, the **Langrangian**, really doesn't work any more. Also, $\mathcal{H}$, the **phase space**, (à la Hilbert space) begins to enter the picture, as does the unitary operator U_t, which denotes time evolution.

So given an L, how would you find U_t? Is there such a functor?

Well, for particles, $\mathcal{H} := \mathbb{L}_2(\mathbb{R}^n)$. So maybe $\mathbb{L}_2(M)$ can be used in general? Can we write something down in this case?

Now a physicist (or an engineer) would like to write an operator on such a space, like:

$$(U_t f)(x) = \int U_t(x, y) f(y) dy$$

...i.e., like a kernel.

It that was then really Dirac and Feynmann who gave a paradigm — the DF paradigm. Do this:

$$U_t(x, y) := \int_{\Omega^t(x,y)} e^{\frac{2\pi i}{h} S(\mu)} \mathcal{D}(u)$$

Many people have tried to make sense of this. (There is no functor because classical mechanics is a limiting case of quantum mechanics, and not vice-versa.)

Let's put forth a baby version of quantum physics.

DIGRESSION *But first, how did the above formula come about? Slit experiments! Common sense addition of probabilities fails in describing the light densities one sees,* **but** *if you use complex amplitudes, then things work out.* ⋄

Let $T_\lambda(f)$: Volumes on $M \longrightarrow \mathbb{R}$. Then we can set $T_\lambda(f)v = \int_M e^{i\lambda f} v$, where we view the exponent as a distribution from volumes to $\mathbb{R}$. Now let $\lambda \to \infty$. We can then look at critical points again!

THEOREM 1. $|T_\lambda(f) \cdot \varphi| < \frac{c_k}{\lambda^k}$, *for all* $k \in \mathbb{N}$, *if the support of* φ *does not contain any criticial points of* f. ■

Here, the techniques of asymptotic expansion begin to appear...

So if you're in dimension N now, and have non-degenerate critical points, you have

$$T_\lambda(f) \sim \sum_p e^{i\lambda(f(p) + \frac{\pi}{4}\sigma)} \frac{1}{\lambda^{n/2}} \sum \delta_p \left\{ 1 + \frac{1}{\lambda} \sum a^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha} \right\} \frac{v}{|\det \frac{\partial^2 f}{\partial x^i \partial x^j}(p)|^{1/2}},$$

where σ is the signature of $[\frac{\partial^2 f}{\partial x^j \partial x^i}]$ (thus playing the role of the difference between positive and negative eigenvalues), a^α is determined by $(\frac{\partial^2 f}{\partial x^j \partial x^i})_p^{-1}$, and we make use of the formalism of **Feynman diagrams**.

This is what all physicists do when they do perturbation theory, but they work in infinite dimensional spaces.

CHAPTER 3

Five Lectures on Equivariant Cohomology

Lecture 1: From Electrical Circuits to Cohomology

Editor's Note: *This lecture was introduced by Professor Roderick Wong, Dean of the Faculty of Science and Engineering, and Professor of Mathematics, at City University of Hong Kong.* ◇

Let me first reflect on what brought me to topology: an applied problem. At Princeton, Herman Weyl was working on Hodge theory. Oddly enough, in its "baby" form in low dimensions, one arrives at circuit theory.

Here is the applied problem: Compute the **effective** resistance of an electrical network. That is, one is given a graph whose edges correspond to resistors, and whose vertices correspond to their interconnections, and then one fixes two vertices to be the + and − connections to a power source.[1] Here, the solution is already like topology...

First, orient each edge at will. Letting $\{E_\alpha\}$ and $\{V_i\}$ respectively denote the set of edges and vertices, we then define the vertex-edge adjacency matrix of the network:

$$[-V_i, E_\alpha] = \begin{cases} +1 & V_i = \text{head of } E_\alpha \\ -1 & V_i = \text{tail of } E_\alpha \\ 0 & \text{else} \end{cases}$$

What would a topologist do now?

Well, first let $C(V)$ (resp. $C(E)$) denote the space of real valued functions on the vertices (resp. edges) of our network. Then the preceding matrix describes a linear operator from $C(E)$ to $C(V)$.

More precisely, let $\varphi \in C(V)$ (resp. $\psi \in C(E)$), and let us introduce the following definitions.

DEFINITION 1. *Let $\delta : C(E) \longrightarrow C(V)$ be defined by*

$$\varphi \mapsto \sum_i [-V_i, E_\alpha]\varphi(V_i).$$

So $\delta\varphi(E_\alpha) = \varphi(E_\alpha) = \varphi(\mathrm{Head}(E_\alpha)) - \varphi(\mathrm{Tail}(E_\alpha))$. ◇

DEFINITION 2. *Let $\partial : C(V) \longrightarrow C(E)$ be defined by*

$$\psi \mapsto \sum_\alpha [V_i, E_\alpha]\psi(E_\alpha). \qquad \diamond$$

[1]Electrical engineers traditionally call graphs **networks**, vertices **nodes**, and edges **branches**.

Clearly, we have that ∂ and δ are duals.

Note now that **Kirchoff's First Law** becomes $i_{ext} = \delta i_{int}$ — that is, the current going into a vertex equals the current going out. Similarly, **Ohm's Law** becomes $G\delta e = i_{int}$, where G is an electrical quantity called conductance and e is the voltage difference between two given vertices.

Now we'd like to solve these equations for G and i_{int}, but the matrix for $\delta^* \times \delta$ is singular.

THEOREM 2. *To find the resistance, R_α, across a branch E_α is exactly*

$$R_\alpha = \frac{\partial}{\partial g_\alpha} 1 + g\psi$$

where

$$\psi = \sum_{T \text{ a spanning tree}} G(T)$$

and

$$G(T) = \prod_{E_\alpha \text{ an edge in } T} g_\alpha. \qquad \blacksquare$$

My doctoral thesis aimed to prove all this, but Kirchoff had already proved all this 200 years earlier! Nevertheless, at Princeton, I learned that all this could be generalized considerably.

In particular, let K be a simplicial complex (a union of simplices intersecting only in simplices of lower dimension, etc.). Let us now forget the g's above, but remember the incidence relations. Ordering the vertices, one naturally gets

$$C^0(K) \xrightarrow{\delta} C^1(K) \xrightarrow{\delta} C^2(K) \xrightarrow{\delta} \cdots$$

where δ is given naturally by our more general incidence relations. The dual of δ, which we will call ∂ as before, gives a sequence of maps going in the reverse direction.

You then get $\delta^2 = 0$ (and similarly $\partial^2 = 0$). Now we get something we didn't see earlier...

$$H^i(K) := \mathrm{Ker}\delta/\mathrm{Im}\delta$$

which is a finite dimensional vector space

THEOREM 3. 1. *$H^i(K)$ are topological invariants of K. (This was really only proved in the 20's.)*
2. *In fact, the $H^i(K)$ are invariant under homotopy equivalance.* $\blacksquare$

DEFINITION 3. *Let $f : K' \longrightarrow K$ and $g : K \longrightarrow K'$ be continuous maps. We then say that they are* **homotopy inverses** (*resp.* **homeomorphisminverses**) $\Longleftrightarrow$ *both $g \circ f : K' \longrightarrow K'$ and $f \circ g : K \longrightarrow K$ are homotopic (resp. homeomorphic) to the identity map* 1.

What about H^0 and H^1?

Well, *dim* H^0 just says how many "boquets" there are in one's complex. This in fact relates to our old problem...

Old Theorem *We have*

$$\sum(-1)^i dimH^i(K) = \sum(-1)^i dimC^i(K).$$

In particular, for a graph, $dimH^0 - dimH^1 = \#v - \#edges.$
The left-hand side of the above equality is known as the **Euler Characteristic** of the complex.

Definition 4. *Define* B^{q+1} *be the image of* C^q *under* δ = *coboundaries* $C^q \xrightarrow{\delta} B^{q+1}$, *i.e., consider the following two exacts sequences implicitly defining cohomology*

$$0 \longrightarrow Z^q \longrightarrow C^q \xrightarrow{\delta} B^{q+1} \longrightarrow 0$$

Since $\delta^2 = 0$

$$0 \longrightarrow B^q \longrightarrow Z^q \longrightarrow H^q \longrightarrow 0$$

where Z *denotes the group of cocycle*

Corollary 1. *(of basic exact sequence facts)*
$\dim Z^q = \dim B^q + \dim H^q$ *and* $\dim C^q = \dim B^{q+1} + \dim Z^q$

Now let
$Z_T = \dim Z^q T^q$
$C_T = \dim C^q T^q$
$B_T = \dim B^q T^q$
$H_T = \dim H^Q T^q$
Then,
the first exact sequence $\Longrightarrow C_T = Z_T + \frac{B_T}{T}$
the second exact sequence $\Longrightarrow Z_T = B_T + H_T$
So, $C_T - H_T = \frac{B_T}{T}(1+T)$
In particular, $C_T - HT = (1+T)Q(T)$ where Q is a polynomial with all coefficients non-negative —— $T = -1$ gives result.
This is very nice for the union of two complexes:
$X = A \cup B$

$$X(X) = X(A) + A(B\_X(A \wedge B)$$

(Mayer-Vietoris property)

Now we can play games...
The Mayer-Vietoris sequence starts from

$$M \longleftarrow A \amalg B \; \overset{\longleftarrow}{\underset{\longleftarrow}{}} \; A \cap B$$

(there are two ways to embed intersection) and gives us the following exact sequences valid for all dimensions:

$$H^q(M) \xrightarrow{r_1+r_2} H_q(A) \oplus H^q(B) \xrightarrow{r_{12}^1-r_{12}^2} H_q(A \wedge B)$$

$$H^{q-1}(M) \longrightarrow H^{q-1}(A) \oplus H^{q-1}(B) \longrightarrow H_q(A \wedge B)$$

The exactness in the middle is easy to check.

Now, the clever thing is that there's a way to go up in dimension.

Assertion: *(a conglomeration of 60 easy lemmata from linear algebra)*
The induced long sequence of maps is exact!!
This allows you to compute the cohomology of spheres.

The modern (40's–50's) development:
Eilenberg & Steenrod found **axioms** for cohomology:

1. Functorial (contravariant)

$$x \rightsquigarrow H^q(x) \text{ and } x \xrightarrow{f} y$$

   implies

$$H^*(x) \xleftarrow{f^*} H^*(y)$$

2. Homotopy invariance
3. Mayer-Vietoris for open sets
4. $H^*(pt) = \mathbb{R}$ in *dim* 0.

The Eilenberg-Maclane axioms, for any resonable space, will imply that **any** cohomology theory will give you the same answer.

Next Time: Taking ideas from homology theory to De Rham theory. Relating Chern-Weil characteristic classes to give an equivariant theory.
The main task will be to see how the above axioms work for the DeRham theory.
μ (C^∞-manifolds)
$\Omega^*(\mu)$ = DeRham complex
where $\omega^0(\mu) = C^\infty$-functions
$\Omega^1(\mu)$ = 1-form
$\Omega^2(\mu)$ = 2-forms

E.g.
Ω^0 is the space of functions on points.
Ω^1 should be functions on curves, but how do you make sense of this? Well, try $\int_\mu adx + Gdy$
Similarly, for surfaces, try $\int adxdy + bdxdz + cdydz$. **Claim:** One is naturaly led to an analogue of δ! (We reach it wrongly as the gradient.)

Lecture 2 (March 11, 1999)

Last time: A bit of network theory.
Recall that we had a complex K which was a union of simplices, $C^k(\Delta)$:= real valued functions on the k-simplices.

$$0 \longrightarrow C^0(k) \xrightarrow{\delta} C^1(K) \xrightarrow{\delta} C^2(K) \longrightarrow \cdots \longrightarrow 0$$

Cohomology measures how much the sequence were fails to be exact:

$$H^i(K) := \mathrm{Ker}\delta/Im\delta|_i$$

i.e., we keep track of the most interesting solutions of $\delta = 0$.
Remarkable Consequence:

1. Topological invariance (first proof due to Kabler in the 1920's) But in the 40's, with Bott at Princeton, these old simplicial methods began to be too cumbersome. Thus entered the Haupvermutung.
   Streamlined (1940's) An abstract cheat cleaned up by Eilenberg and Steenrod. Setup: Take standards (like the bureau of standards in Washington)

   Definition 5. $X_k(X) = Map(\Delta_k, X)$
   where X is our space, and $Map(X, Y)$ denotes the space of all continuous maps from X to Y.

   Now, define C^K = All functions from $S_k(X) \longrightarrow \mathbb{Z}$ (or any coefficient ring). So $C^0(X)$ = All $\mathbb{Z}$-valued functiones on X.
   $C^1(X)$ = All $\mathbb{Z}$-valued functions on curves on X.

   $$C^0(S) \xrightarrow{\delta} C^1(S(x))$$

   Definition 6. $\delta f(u(T)) = f(u(1)) - f(u(0))$

   ...Same proofs, but with a **huge** complex. The **singular** complex of X.
   You get

   $$C^0(S) \longrightarrow C^1(S) \longrightarrow C^2(S) \longrightarrow ...$$

   g and $\delta^2 = 0$ and $H^0(X) = \mathrm{Ker}/Im$ etc. It's built to be functional! (Easy to check!)
   So this is a function: $X \rightsquigarrow H^*(X,\mathrm{Z})$ which is contravariant. What was main involved was that this was correct in finite dimensions. One then simpl checks the axioms.
2. $H^q(pt, \mathbb{Z})$ is $\mathbb{Z}$ or 0 according as q is 0 or not.
   Proof:
   (Only one for now)
   what is $S_k(pt)$? It is a constant !!
   So for all k, $C^k(S(X)) = \mathbb{Z}$

Now, we can get what we want...

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{Id} \mathbb{Z} \xrightarrow{0} \dots$$

REMARK 1. *The preceding construction could have also been done with cubes (like Serre did), but the proofs would have been hairier.*

3. Homotopy invariance: If you have $X \xrightarrow{ft} X = x \times I \longrightarrow Y$. Now you have to product the simplices with I and sudivide. But then you're done.. (Just combinatorics with simplices...)
4. Mayer-Vietoris: $X = U_1 \cup U_2$

$$X \longleftarrow U_1 \amalg U_2 \overset{\longleftarrow}{\underset{\longleftarrow}{}} U_1 \cap U_2$$

$$H^*(x) \longrightarrow H^*(U_1) \oplus H^*(U_2) \longrightarrow H^*(U_1 \cap U_2) \longrightarrow H^*(X)$$

By Mayer-Vietoris Property, this is not hard to show.

Therefore, the finite dimensional construction gives the right answer. (The proofs was rather technical barycentric sublic, sufficiently small simplices.)

NOW: In the case of manifolds, there is a nice subcomplex which greatly simplifies things.

Manifolds:

We have an atlas:

i.e. $U_\alpha; X^1_\alpha, \dots, X^n_\alpha \qquad P \longmapsto (X^1_\alpha(P), \dots, X^n_\alpha(P))$ where $U_\alpha \in \mathbb{R}^n$

A C^∞ -atlas $\Leftrightarrow$ or $U_\alpha \cap U_\beta, [-\frac{\partial x^i_\alpha}{\partial x^J_{beta}}] =: J_{\alpha\beta}$ is well-defined and nosingular everywhere. i.e., $J_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(n)$ is constant.

where $J_{\alpha\beta}$ is the Jacobian of the atlas.

$F_1 = C^\infty$ functions on X. then $f|_{U_\alpha}$ is C^∞ in X^1_α, ..., X^n_α.

SingularTheory	**DeRham**
$f : X \longrightarrow \mathbb{R}$	$f : X \longrightarrow \mathbb{R}$
$where X is constant$	$where X \in C^\infty \longrightarrow \mathbb{R}$

Consider $df = \frac{\delta f}{\delta X^i} dx^i$

since $df = 0$, that is f is locally constant.

On a connected manifold, locally constant implies globally constant. The dim(Ker$d$) on a C^∞ - manifold = # of correlated compos

De Rham Complex: $\omega^0(\mu)$ $\omega^1(mu)$ Given $w = \sum a_i dx^i$ and μ:1 $\longrightarrow \mathbb{R}^n$, then it makes sense to take $\int_m u a_i dx_i$.

Technically, $\mu^x a_i d\mu^x x^i = \int_\mu a_i dx^i$

In 2 dim:

So defining cohomology is like solving differential equations.

$$H_D R^i(\mu) := \mathrm{Ker}(d)|^i / Im(d)$$

$$df := \frac{df}{dx^i} dx^i$$

Normally, this is contravariant.

Checking the axioms is not so bad (done in Bott and Tu).

Remark 2. *It is easier than trying some kind of embedding as a subcomplex.*

Let's think a bit about basis geometry.

$$M_1\{U_\alpha, J_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(n)\}$$

$J_{\alpha\beta}$ says something about changing coordinate.
e.g.

$$\begin{aligned} df &= \frac{\partial f}{\partial x^i_\alpha} dx^i_\alpha \\ &= \frac{\partial f}{\partial x^i_\beta} dx^i_\beta \end{aligned}$$

(1)

on $U_\alpha \cap U_\beta$, via the chain-rule,

$$\frac{\partial f}{\partial x^i_\alpha} = \frac{\partial f}{\partial x^J_\beta} \frac{\partial x^J_\beta}{\partial x^i_\alpha}$$

But $\frac{\partial x^J_\beta}{\partial x^i_\alpha}$ are just the curves of $J_{\alpha\beta}$! So we can think of ∂f as a twisted function. (as our frofathers did)

Definition 7. *Let*

$$p : GL(a) \longrightarrow GL(m)$$

be any rep. (i.e. any homotopy). Then a twisted function of type p is an assignment

$$\alpha : f_\alpha : U_\alpha \longrightarrow \mathbb{R}^m$$

Now on $U_\alpha \cap U\beta$, we have $p(J_{\alpha\beta} f_\beta = f_\alpha$.
ie. Tensor of type p on m. (Ordinary functions: $P :\longrightarrow R_(1)^*$)

So the ω^i are the twisted functions associated to $\Lambda^i \mathrm{GL}(n)$ $(\vee \longrightarrow \wedge^i \vee)$

Note that the 2^{nd} derivative are symmetric $\Longrightarrow$ extension powers.
This is a modernization of Eisenhardt's book. (In the old framework, a tensor was something which was just storms!)
Now...

Use (J, ρ) to make a new space
A vector bundle!

$$E(\rho) \xrightarrow{\pi} M \text{ and } \pi^{-1}(\rho) = \mathbb{R}^m$$

where π is canonical

So now you can speak of **sections** s.
(S is said to be a section if and only if $\pi \dot{s} = 1$.)

We eventually see that twisted fuctions of type $\beta \Longrightarrow r(E(\rho))$ (which is the sections of $E(\rho)$)

Translation? We can start with atlas and multiply by $\mathbb{R}^m$.

$$U_\alpha \times \mathbb{R}^m \xrightarrow{\pi} U_\alpha$$

where π is the ordinary projection onto the 1^{st} factor.
A section is just $s(x) = (s, f(x))$ that is "a function."

Now glue

$$U_\alpha \times \mathbb{R}^m \xrightarrow{\pi} U_\alpha \text{ and } U_\alpha \times \mathbb{R}^m \xrightarrow{\pi} U_\beta$$

Then we have..

$$(x, v_\alpha) \overset{x \in U_\alpha \cap U_\beta}{\longleftarrow = \longrightarrow} (x, \rho(J_{\alpha\beta}^{-1}(x))v_\beta)$$

So tensors $\leftrightarrow$ sections of a bundle.
Let us now finally compare some DeRham cohomology.

Example: Take the circle: Start with $\mathbb{R}$...

$$\Omega^0(\mathbb{R}) = C^\infty \text{ - functions}$$
$$\Omega^1(\mathbb{R}) = f(x)dx$$
$$\Omega^2(\mathbb{R}) = 0$$

So

$$\Omega^0(\mathbb{R}) \longrightarrow \Omega^1(\mathbb{R})$$
$$f \mapsto f'dx$$
$$\text{Ker} = \text{constant functions} = \mathbb{R}$$
$$\to H^0(\mathbb{R}) = \mathbb{R}$$
$$H^1 = C^\infty / C\frac{\partial}{\partial x}C^\infty = 0$$

Homework:

Try, for $\mathbb{R}^n$, to get the Poincare lemma: This lemma, which is frequently stated without proof asserts that

$$H^x_{\alpha\beta}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & 0 \\ 0 & x > 0 \end{cases}$$

Now let's do the circle...
($\mathbb{C}^0$ = Periodic C functions)

Old argument:
$H^0(S^1) = \mathbb{R}$ via the same argument. **But**

$$d\int_0^x g dx \Leftrightarrow \int_0^1 g dx = 0.$$

i.e.

$$\int_{S^1} H^1(S^1) \longrightarrow \mathbb{R}$$

New argument: (Fourier series!) (Really $\mathbb{C}$ version)

$$f = \sum a_i e^{in\theta}$$

DEFINITION 8. $\{a_n\} \longrightarrow \{ina_n\}$ *So* Ker *is again constant functions. Im? = All a's, when* $a \neq 0$. *So* C Ker = *Constant functions. Hodge theory in the simplest case. So* $H^1(S^1) = \mathbb{R}$

Next time: A bit more DeRham theory.
Chevalley, in the 40's, defined tensors by starting with the C^∞ functions, and then going to an algebraic definition of the tangent space.

$$X_p : F \in \mathbb{C}^\infty \longrightarrow \mathbb{R}$$

$$X_p(f\dot{g}) = f[p] \times \dot{g} + g(p) \times f$$

TM: The tangent bundle.
Nows take T*M to get the cotangent bundle (which is the best way).

Lecture 3 (March 18, 1999)

Last time, we had some propaganda for the Brauer complex...

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \longrightarrow \cdots \longrightarrow \Omega^k(M)$$

The above sequence is of order 2.

$$H^i(M) = \mathrm{Ker}d/\mathrm{Im}d$$

(Beauty: In the C^∞ category there is no need to chop things up into simplices!)

EXAMPLE 4. *in* $\mathbb{R}^2$

$$d(ydx - xdy) = dydx - dxdy = 2dydx$$

So the latter is a **closed** *for* d

Now, let's look at $\mathbb{R}^2 \setminus \{\vec{0}\} \ldots$
Consider

$$\frac{xdy - ydx}{x^2 + y^2}$$

which is a C^∞ 1 - form on $\mathbb{R}^2 \setminus \{\vec{0}\}$.
Then,

$$d(\frac{xdy - ydx}{x^2 + y^2}) = 0$$

So this somehow represents the cohomology. Here's a technique: Find a circle along which $\int (lastform) \neq 0$.

You then get

$$\begin{aligned} \mu^x X &= \cos\theta \\ \mu^x Y &= \sin\theta \\ \Longrightarrow \mu^x W &= \cos\theta\cos\theta d\theta + \sin^2\theta d\theta \\ &= d\theta \end{aligned}$$

(2)

So

$$\int_0^{2\pi} \mu^x w = \int_0^{2\pi} d\theta = 2\pi.$$

Now if $w = df$ then

$$\int_0^{2\pi} \mu^x w = \int_0^{2\pi} d\mu^* f = \mu^x f|_0^{2\pi} = 0$$

So w is not closed.

$\Longrightarrow$ the basic trick: Look at Ω or $M \setminus V$. Look for things which blow up or vanish. There is a theorem of Grothendieck to this effect.

Another Example: In $\mathbb{R}^3 \setminus 0$

Take

$$\frac{xdydz - ydxdz + zdxdy}{(x^2+y^2+z^2)^{3/2}}$$

After integrating, you eventually get $4\pi = \int \mu^x W$.

First-example revisited

$x^2 + y^2 = 1$ is the circle too.

Now think of $f = x^2 + y^2$ and the **level** $f = 1$.

Well,

$$df = d(x^2+y^2) = 2xdx + 2ydy$$

$$x^x(xdx + ydy) = 0$$

i: $S^1 \longrightarrow \mathbb{R}^2$

$U = \{x \neq 0\}$

Define θ or $U \cap S^1 = \frac{i^*dy}{i^*x}$

$V = \{y \neq 0\}$

Well, since $\frac{i^*dy}{i^*x} + \frac{i^*dx}{i^*y} = 0$

we obtain ...

$$\theta = \begin{cases} \frac{i^*dy}{i^*x} & U \cap S^1 \\ -\frac{i^*dx}{y} & V \cap S^1 \end{cases}$$

Now, $\mu^x\theta = \frac{\cos\theta d\theta}{\cos\theta} = d\theta$

General Direction (a warm up to grander things):

If M is a compact orientable manifold, then there is a notion of integration.

$$\Omega^n(M) \overset{S_{[-m]},\pi_x}{\longrightarrow} \mathbb{R}$$

($\mu \overset{\pi}{\mapsto} pt$; Grothendieck would say....)

In advanced calculus, we're taught evaluating functions, taking parallelepipeds, etc... $\iint f(x,y)\,dx, dy$. But what we really need is that when you pull back, you get $|J_{ac}|$ as your normalizing factor.

Now,looking are the differential forms...

$$f(x,y)dxdy \mapsto J_{ac}\mu^x$$

Note the **new** ambiguity!
So do the following ...

1. Find an atlas for M with $\det(J_{\alpha\beta}) > 0$ for all α, β.
2. Lemma: there exists arbitrary a, b fine partitions of 1 i.e. there exists φ such that $\sum \varphi$=1 and $\varphi > 0$

You get an $\int$ defined up to sign..
i.e.

$$\int_{[-M]} : \Omega^n(M) \longrightarrow \mathbb{R} \longrightarrow 0$$

We eventually get..
Stoke's Theorem:

$$\int_{[-M]} dy = 0$$

(reduce to the case of $\mathbb{R}^n$ where the functions have compact support.)
so,

$$\int : H^n M \longrightarrow \mathbb{R}$$

Now..
Poincare Duality: (which Poincare never actually proved) There is a pairing

$$\langle\cdot,\cdot\rangle : H^q M \otimes H^{n-q}(M) \xrightarrow{\int_{[-M]}} \mathbb{R}$$

So you also get a pairing on $(\oplus H^1(M)) \otimes (\oplus H^1(M))$

Is it nondegenerate?
(Sometimes you see written that $\langle\cdot,\cdot\rangle$ is a "dual" pairing...
i.e., Fixing ψ gives you the linear map..

$$w \mapsto \langle w, \psi\rangle$$

Thus ψ (via $\langle,\rangle$) can be considered as an element of $H^{n-P}(M)^*$!)

$$\text{Poincare Duality} \Longrightarrow \textbf{all} \text{ elements of } H^{n-p}(M)^*$$

arise this way!

Remark 3. *Poincare did not think in this context so much as combinatorially.*

1. Lefschetz used "cells" to prove this in the general simplicial (CW?) case.

2. More modern: Just extend to-compact, then reduce to case of $\mathbb{R}^n$ (using Mayer Vietoris...) To do this, start up function of compact support. Then

$$M\Omega^*(M) \otimes \Omega^*_c(M) \longrightarrow \dots$$

3. **Method:** choose a Riemann structure on M....
   i.e. The tensor on M:

$$dx^2 = g_{iJ}dx^i dy^J$$

(using Einstein's sum convention). This introduces an inner product or all the tangent, cotangent etc. bundles.

We can now use orthonormal **frames**, i.e., a set of f-forms $\theta_1, \ldots, \theta_n$
They span the space of one forms in U.
We can now speak of orthonormal frames. $\forall i \ldots$
Let's look at $n = 2$ (Riemann surfaces).

Definition 9.

$$*\theta^1 = \theta^2$$

$$*\theta^2 = -\theta^1$$

More generally, let

$$\begin{aligned} *\theta^1 &= \theta^2 \wedge \cdots \wedge \theta^n \\ &\vdots \\ *\theta^n &= (-1)^{n-1}\theta^1 \wedge \cdots \wedge \theta^{n-1} \end{aligned}$$

Fact: * is well-defined and can be interpreted as an isomophism of $\Omega^k \longrightarrow \Omega^{n-k}$.

Is it independent of d?

Now ...

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n$$

Definition 10. $\forall \theta, \varphi \in \Omega^k$,

$$< \varphi, \psi >:= \int \theta \wedge *\varphi$$

(so each Ω^i is now a pre-Hilbert space, i.e., it has an inner-product.)

Is there a dual? **YES!**

$$d^* := \pm * dx$$

where the $\pm$ sign is complicated — an irritant which gets you into spin.

Now we use **Hodge Theory** to prove Poincare Duality...

$$(d + d^*)^2 = d^2 + \underbrace{dd^* + d^*d}_{\text{This is the Laplacian}} + d^*d^2$$

At each Ω^k, we have $\Box_k = dd^* + d^*d$ at k.

Furthermore,

$$d\Box_k = \Box_k d$$

$$d^x\Box_k = \Box_k d^*$$

Now, let's recall how we compared $H^x(S^1)$ via Fourier series...
Well, on the circles we had the operator

$$-\frac{d^2}{d\theta^2}$$

This has eigenspace e^{inx}..
The idea of Hodge theory is to decompose each space Ω^k relative to $\Box_k$....

In 1959, there was no way at doing this. but in the 60's, various results in differential equations finally were derived (but this was known to De Rham.).

THEOREM 4. *$\Box_k$ is elliptic:*

$$\Omega^k \longrightarrow \Omega^k$$

$$\Omega^k = \oplus\Omega^k_\wedge$$

where $\Omega^k_\wedge$ is an eigenspace.

$$\Omega^k_\wedge = \{e \in \Omega^k | \Box_k\varphi = \lambda\varphi\}$$

where $\Omega^k_\wedge$ is elliptic $\Longrightarrow$ finite dimension! and $d\varphi \in \Omega^{k+1}_\lambda$.

Ah! So instead of studying something horrible at $\dim \Omega^k$, we can reduce to some nice finite dimensional $\Omega^0_\lambda \rightleftharpoons \Omega^1_\lambda \ldots$. This gets you **harmonic** forms: Those that satisfy the 2^{nd} order differential equation $\Box_k\varphi \equiv 0$

$$\Omega^0_0 \rightleftharpoons \Omega^1_0 \ldots..$$

Trick: The $\Omega^i_0$are exactly cohomology and there is no cohomology in the higher λ's because, for $\Lambda \neq 0$,

$$d_\lambda d^*_\lambda + d^*_\lambda d_\lambda = \lambda$$

implies

$$d_\lambda \frac{d_\lambda}{\lambda} + \frac{d_\lambda}{\lambda} d_\lambda = 1$$

Given a cocycle φ_k (so $d\varphi_\lambda = 0$), we get

$$(d_\lambda \frac{d_\lambda}{\lambda} + \frac{d_\lambda}{\lambda} d_\lambda)\varphi = \varphi$$

$$d_\lambda (\frac{d^x_\lambda}{\lambda})\varphi = \varphi$$

Ah! So cocycle = coboundary!

So, once you've chosen a Riemannian structure, in every cohomology class there is a unique harmonic form. Thus

$$H^*(\mu) = H^*(\mu)$$

where $H^*(\mu)$ is the harmonic form on μ. (This harks back to electrical networks: the resistors are what defined the pairing $\langle \cdot, \cdot \rangle$)

We have thus arrived at Hodge theory. Interestingly, it made Hodge famous, but wasn't proved till 1999. In fact, one could argue that there were better methods by Hermman in the 60's.

Next time, we'll speak about cohomology of Lie groups, where this (not cutting up into triangulations) is the best way to go. In equivariant cohomology , you **mix topology of** M and the **the axioms of** G. This gives the conceptual link between differential geometry and topology of characteristic classes.

Lecture 4: Equivalent Cohomology

Last time: We talked about the effect of a Riemannian structure and the Hodge theorem:

$$\dim H^{n-k}(M) = \dim H^k(M)$$

Get a d^* from d going the opposite way.
Let's now use this to explain Maxwell's Equations:

In $\mathbb{R}^4$, $dF = 0$ and $d^*F = 0$. That is

$$dd^* + d^*d = \square$$

is no longer an elliptic equation — instead, it is a wave equation. For $F \in \Omega^2(\mu)$, Yang-Mill's theory is a general way to extend this to n-dimensions. Let us go to a different subject now — a place where differential forms are absolutely decisive: Lie Groups.
M (a group which is also a manifold.)

$$MxM \xrightarrow{\mu} M$$

where M is a C^∞ manifold, and μ is smooth.

Example 5. $(\mathbb{R}^4, +)$...

$$\textit{Let's consider } \mathrm{GL}^n(\mathbb{R})$$
$$O(n) \subset \mathrm{GL}^n(R)$$
$$SO(n),\ U(n)...$$

$$SP(n) := A/A^TJA = J, where J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \diamond$$

Amazing Fact: Amongst compact Lie Groups, that's all there is... except for 5 **exceptional** ones.
So how do you at last compute the cohomology? Let's consider the set G and g is in g,

$$Lg : G \longrightarrow G$$

$$x \mapsto gx$$

Let's speak of forms that satisfy,
for all g,

$$L_g^* w = W$$

That is, **left-invariant forms**. You get

$$LI\Omega^*(G) \subset \Omega^*(G)$$

Easy Facts:

1. invariant functions are constant.
2. $\dim LI\Omega^*(G) = \dim \wedge^n T^*_{\not{k}} G$, which is finite.
   Since

$$dL^*g = L^*gd$$

$$\implies dL^*gw = dw$$

$$\implies L^*gdw = 0$$

So

$$d^2 = 0$$

Choose a basis for the $LI\Omega^*$ ... θ:
We then get

$$d\theta_i + \frac{1}{2} C_i^{Jk} \theta_J \wedge \theta_k = 0$$

where the structure constants satisfy $C_i^J k = -C_i^k J$
WHY? — It satisfies the Jacobi identity...
(So Classifying finite dimensional complexes on a Lie group? = Classifyinig Lie groups/algebra)

EXAMPLE 6. $G = \mathbb{R}$

$$x$$

$$L_a^x x = x + a$$

$$L_a^x dx = d(x+a) = dx$$

so dx is a left-invariant form... Therefore cdx = All left-invariant forms and G=$\mathbb{R}^*$

$$x \neq 0$$

$$L_a^* x = ax$$

So $L_a^* dx = adx$ Thus $\frac{dx}{x}$ is an invariant differential. In many formulas you'll find $\frac{dx}{x^\alpha}$ because of relations left-invariant of the multiplicative group.

Note that in all case, we've had $d\theta = 0$. Now what is the cohomology of our new smaller complex? For $\mathbb{R}$

$$H^*(Inv\mathbb{R}) = \begin{cases} \mathbb{R}; & 0 \\ \mathbb{R}; & 1 \end{cases}$$

which is not right, but if we translate by the units we get $\mathbb{R}/\mathbb{Z} \cong S^1$, and by the left-invariant forms on $\mathbb{R}$! (The Lie algebrae of two distinct lie groups need not be distinct.)
Now let's try GL($\mathbb{R}$)... (GL(n))
Let $G = \mathrm{GL}(n)$, but also think of the **entries** of any element as a function on G. Better to

think of g as the matrix valued function on G.
So

$$dg := [dg_{iJ}]$$

Now what is $L_A^* g$?
Well, just $A \cdot g$

So $dL_A^* g = Adg$ and the invariant forms are just

$$\Theta = g^{-1} dg$$

Now how do we find the structure equation? i.e., $d\theta$=?
Well, since

$$g^{-1} \cdot g = 1$$

let us differentiate implicitly:

$$(dg^{-1})g + g^{-1}dg = 0$$

$$\Downarrow$$

$$dg^{-1} = -g^{-1}dgg^{-1}$$

So

$$\Theta = g^{-1}dg \Longrightarrow d\Theta = -g^{-1}dgg^{-1}dg+)$$

$$\Downarrow$$

$$d\Theta + \Theta^2 = 0$$

(The structure equations of the full linear algebraic Lie group.) They give you all the structure constants in the basis. We can then compute cohomology, since we now know how to write any differential form in terms of this basis.

Example 7. GL(2)

$$\begin{array}{cc}
\textit{Basis for } \Omega^1 & \textit{Basis for } \Omega^2 \\
\theta_1 1 \quad \theta_1 2 & d\theta_1 1 + \theta_1 2 \theta_2 1 = 0 \\
\theta_2 1 \quad \theta_2 2 & d\theta_2 2 + \theta_2 1 \theta_1 2 = 0 \\
 & \Downarrow \\
 & d(\theta_1 1 + \theta_2 2) = 0
\end{array}$$

So then you get

$$H^*(\mathrm{InvGL}(2)) \quad = \quad \begin{array}{cc}
\mathbb{R} & 0 \\
\mathbb{R} & 1 \\
0 & 2 \\
\mathbb{R} & 3 \\
\mathbb{R} & 4
\end{array}$$

A Hermann Weyl trick is useful in the above: $\mathrm{GL}(2) = O(2) \times$ *(Lower Triangular Matrices).*

What about the other groups?

$$O(A) \subset \mathrm{GL}(n, \mathbb{R})$$

You can restrict forms....

$$2^*\Theta \text{ from } \Theta \text{ in } O(n)$$

or

$$O(n)$$

e.g., what was the defining equation of $O(n)$? $gg^T =$? Well,

$$2^*\Theta = g^T dg$$

But $d(g^T g) = dg^T g + g^T dg = 0$. Since $(i^g(\Theta))^T = dg^T g)$ this implies $\Theta^T + \Theta = 0$. So for SO(3)...
Lie Algebra $\Longrightarrow$ $3x3$ skew-symmetric matrices...
Let's compute

$$\begin{pmatrix} \ldots & \theta_{12} & \theta_{13} \\ 0 & \ldots & \theta_{23} \\ 0 & 0 & \ldots \end{pmatrix}$$

three generators

$$d\theta_{12} + \theta_{13} \wedge \theta_{32} = 0$$

$$d\theta_{23} - \theta_{12} \wedge \theta_{23} = 0$$

$$d\theta_{13} + \theta_{12} \cdots \theta_{23} = 0$$

So d maps all the 1-forms onto all the 2-forms.

$$\begin{array}{ccc} H^* & \longrightarrow & \mathbb{R} \\ & & 0 \\ & & 0 \\ & & \mathbb{R} \end{array}$$

(The last term is $\mathbb{R}$ since at the top you have $\theta_{12}\theta_{13} \cdots \theta_{23}$ and this is not d of anything (since θ_{iJ} is always d of something.))
This happens to be the cohomology of $\mathbb{P}^2_{\mathbb{R}}$, and is how the old Cartan must have thought.

Theorem 5. *Suppose a Lie group G acts on a manifold M*

$$(GxM \longrightarrow M)$$

Then we can speak of the G-invariant forms on M If G is compact and connected, then

$$H^*(M) \cong H^*(Inv_G\Omega^*(M))$$

This result was used by Cartan and Weyl. Two principles involved:

1. You can average (by compactness)
2. i.e.

$$Inv_G\Omega^g(M) \subset \Omega^2(M)$$

and

$$Inv_G\Omega^g(M) \xleftarrow{\pi} \Omega^g(M)$$

by averaging, i.e.,

$$\pi : \omega \rightsquigarrow \int_G Lg^x \omega dg$$

Also Ω and Inv$G$ commute with d, G Since $\pi \circ i = \nVdash$,

$$\Longrightarrow H^*(Im\Omega^*(M)) \hookrightarrow H^*(M)$$

and is *onto* the invariant part of $H^*(M)$. Now since g is connected, we obtain a homotopy from $g \longrightarrow 1 \Longrightarrow L_g^* = \nVdash$

However, InvΩ need not be finite dimensional — that depends on M/G....

Now we can start....

The program of the 50's and 60's: Study (i.e., compare) actions of Lie groups on a manifold. Usually, we'd like to classify such actions.

Example 8. *(S^1 acting on M.)*

Famous: $|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 = 1$, $S^{2n+1} \subset \mathbb{C}^{n+1}$

$e^{i\theta}$ *acts via* $e^{i\theta}(z_0, \ldots, z_n) = (e^{i\theta}z_0, \ldots, e^{i\theta}z_n)$. *The orbits are circles and there are no fixed points,and the action is* **free**. *The space of orbits is then something quite important.*

$$\begin{array}{ccc} S^{2n+1} & & \textit{The Hopf Fibration} \\ S^1 \downarrow & & \textit{For } S^3 \\ P^A_{\mathbb{C}} & & S^1 \downarrow \\ & & S^2 \end{array}$$

We get a nonzero element of $\pi_3(s^2)$. *So* $\pi_3(S^2) \neq 0$ *(!)*

Now suppose S^1 acts freely on a manifold.

$$\begin{array}{cc} & M \\ S^1 & \downarrow \pi \\ & M/S^1 \end{array}$$

Claim: You can always find a local cross-section if the center is free, i.e., if the action is free, M/S^1 will leave a manifold structure of dimension $n-1$. Furthermore, π always has a local section..

How fo you get the section?

1. Pick a Riemannian metric on M.
2. Average it over $S^1 \longrightarrow S^1$ now acts by isometry.
3. Take a little perpendicular piece (possibly by Riemannian Structure).

Can you make it global? No (in general)
Hopf Fibration: $S^3 \longrightarrow S^2$ has a local cross-section but not a global cross-section.

Question examined by Whitney, Steenrod and Eckmann (via principle S^1 bundles): how many principle S^1-bundles y can we make over x?

i.e. $S^1 \begin{array}{c} y \\ \downarrow \\ x \end{array}$ and $S^1 \begin{array}{c} y^1 \\ \downarrow \\ x \end{array}$ are identified if and only if

$$\begin{array}{ccc} y & \rightleftharpoons^{u}_{x} & y^1 \\ \downarrow & & \downarrow \\ x & \rightleftharpoons & x \end{array}$$

Equivalently, $\chi \circ \varphi = \varphi \circ \chi$

Topologists found:

THEOREM 6. *The isomorphism classes of S^1-bundles over X (say a k-manifold) $\simeq$ homotopy classes of maps: $f : X \longrightarrow \mathbb{P}^N_{\mathbb{C}}$ for N sufficiently large!*

So the Hopf fibration is universal in a certain sense.
Abstract nonsense:

$$\begin{array}{ccc} y & \longrightarrow & S^{2N+1} \\ ??^{-1}_{\pi} \downarrow & & S^1 \downarrow \pi \\ x & \xrightarrow{f} & \mathbb{P}^N_{\mathbb{C}} \end{array}$$

Main theme of these lectures: Always pull-back! So in particularly given $\begin{array}{c} y \\ \downarrow \pi \\ y/S^1 \end{array}$ you then have a map. $f := y/S^1 \longrightarrow \mathbb{P}_{\mathbb{C}}N$ Now $f^*, c_1 \in H^2(y/S^1)$, c_1 is a generator of $H^2(\mathbb{P}^N_{\mathbb{C}})$, where $\mathbb{P}^N_{\mathbb{C}} \cong \mathbb{R}$ and c_1 can be picked to give $\mathbb{Z}$ even (the Poincare dual of a hyperplane.)
Question: How does this look in De Rham Theory?

Lecture 5 (At Chinese University)

We are going to speak about EC (equivariant cohomology), but we will try to introduce it too. At the end, this relates cohomology to homotopy (topology and analysis).

Theme: Topology of a differential manifold can be studied via topology and homotopy invariants, but you can also study things via DeRham theory.

So let's assume $H^*(\Omega^*(M))$ is understood.
Let's look at things from an invariant point of view, then bring things back to the differentiable setting.
Define M be a compact smooth manifold, and G be a Lie group.
E.g. S',$\mathbb{Z}/2\mathbb{Z}$
Now assume M is G-space (i.e.,G acts on M)
Also

$$M \times G \longrightarrow M \text{ etc..}$$

This decomposes M into orbits.

The **naive** quotient just operates set-theoretically.
E.g. $G = S^1$, $M = S^2$, action via rotating about the north pole.
$\Longrightarrow$ naive quotient = closed interval.

Message: The naive quotient is **only** good if the action is free. i.e. $\begin{array}{c} M \\ \pi\downarrow \\ M/G \end{array}$ is a smooth map of manifolds if the action is free.
Topologists studied the case of free actions for compact Lie group acting on compact manifolds and discovered (by Bott, before 1950)...

First Theorem: Preceding statement on

$$M \xrightarrow{\pi} M/G$$

+Interesting phenomenon: The free actions of a compact group is an acyclic space, such that for all n, $\mathbb{P}_n(E) = 0$ (so free actions are hard to find)...
E.g. $\mathbb{Z}/2\mathbb{Z}$ acting on itself... Easy, but disconnected
$\mathbb{Z}/2$ acting on S^2: we get $\begin{array}{c} S^* \\ \downarrow \\ \mathbb{P}^n_{\mathbb{R}} \end{array}$ but S^* has non-zero H.
Points No free quotient of n-dimensional space is contractible.
But now do the following instead:

$$\begin{array}{ccc} S^n & \hookrightarrow & ... \\ \downarrow & & \\ \mathbb{P}^n_{\mathbb{R}} & \hookrightarrow & ... \end{array}$$

$$\lim_{n \longrightarrow \infty} \mathbb{P}^n_{\mathbb{R}} = \mathbb{P}^{\infty}_{\mathbb{R}}$$

.. now you get that $\mathbb{P}^{\infty}_{\mathbb{R}}$ is contractable
This is a **fine** action on a contractable space.

DEFINITION 11. $\mathbb{Z}/2\mathbb{Z} \rightsquigarrow B(\mathbb{Z}/2\mathbb{Z}) = \mathbb{P}^\infty_\mathbb{R}$, $\quad S^1 \begin{array}{c} E \\ \downarrow \\ BS^1 \end{array}$ *In general, $G \rightsquigarrow BG$ where BG is the classifying space for G.*

Now if we have any **free** action

$$\begin{array}{ccc} & & E \\ & & \downarrow \\ M & \xrightarrow{f} & B \end{array}$$

and any map

We can introduce a pull-back:

$$\begin{array}{ccc} f(E) & \longrightarrow & E \\ \downarrow & & \downarrow \pi \\ M & \xrightarrow{f} & B \end{array}$$

$\nearrow \qquad\qquad \nwarrow$

This free action is a dilution of this action.

Question: Is there a **universal** such action?
Theorem taught to Raoul by Steenrod in 1949 is just this.
(These are principal bundles, therefore it is no need to get scared.)

Given any free action G on M, then $M = f^{-1}_{(m)}E$. Furthermore the number of free actions with base B' are in one-to-one correspondence with the homotopy classes of maps $B' \longrightarrow BG$.

Illustration: Notice that $M = f_M^{-1}(E)$,

$$\begin{array}{ccc} M & & E \\ \downarrow & & \downarrow \\ B' & \xrightarrow{f_M} & BG \end{array}$$

given B', there exists a map f_M.
This is an easy theorem..
Here is why, the main idea coming from French school (Cartan, Borel, Serre,...).

DEFINITION 12. *Given two free actions:*

$$\begin{array}{ccc} M & & M' \\ \pi \downarrow & & \downarrow \\ M/G & & M'/G \end{array}$$

then they can be "mixed", i.e., create the mixing diagram.
(Do Eilenberg's principle: do what's the most natural..)

$$\begin{array}{ccccc} M & \xleftarrow{\pi_1} & M \times E' & \xrightarrow{\pi_2} & E \\ \downarrow & & \downarrow \textit{diagonally} & & \downarrow \\ B' & \longleftarrow & M \times_G E & \longrightarrow & B \end{array}$$

Claim:
Inverse image of a point is B, from the right is a copy of E, and by symmetry, inverse image of a point $\in B$ from the left is a copy of M.

NOW, using contractability we can find a section s where s is....

$$\begin{array}{ccccc} M & \longleftarrow & M \times E & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow \\ B' & \underset{s}{\overset{\sigma}{\leftarrow\!\dashrightarrow}} & M \times_G E & \xrightarrow{\tau} & B \end{array}$$

At last! $f_M = \tau \circ s$.

Therefore it is interesting to see the cohomology of BG! WHY? Because every cohomology class of BG corresponds to a homotopy class of maps $B' \longrightarrow BG$, and this is a cohomology class of B' (where we have...

$$\begin{array}{ccc} M' & & \\ \downarrow & & \\ B' & \xrightarrow{f_M} & BG \end{array}$$

for $a \in H^*(BG)$, $f_M^x(a) \in H^*(B')$.) These $u \in H^*(BG)$ are called the **characteristic classes** of the free action.

Pontryagin:
Now you can do the following, put my Riemannian structure on M (all are homotopic, so it's OK.)

and now consider **frames** $\begin{array}{c} F(M) \\ \downarrow . \\ M(g) \end{array}$ $O(n)$ acts naturally on $F(M)$.

You then get a free action over M
$\Longrightarrow$ a map into $H^*(BO(n))$..
$\Longrightarrow$ Pontryagin classes, Pontryagin numbers.
End of the topologists' story.

You can see how much a bundle is twisted by looking at its characteristic classes.
Now... the De Rham picture (done in the 1950's and 1960's by Cartan, Chevalley, and with a big hand from Andre Weil) Completely different view than from topologists.
$G : \Omega^* G \supset \mathrm{Inv}\Omega^* G$ is finite dimensional.
Explicitly.... $\Omega^*(g)$ $d\theta^i + \frac{1}{2} c_{Jk} \theta^J \theta^k = 0$
Alegebra describes the cohomology of G, i.e., $H^*(G) = H^*(\Omega^*(g))$

Now how do you make such an algebra treatment of the classifying space?
Interestingly, the physicists were doing something smilar around the same time!

$$\begin{array}{ll} & E = (\text{acyclic} K(\Omega)) \\ \textbf{Want:} & \downarrow \\ & BG \end{array}$$

Answer: Weil algebra: $W(g) = \wedge^* g^* \otimes S^*(g^*)$

Exterior variables $\longrightarrow \theta^i$ for invariant forms.
Symmetric variables $\longrightarrow u^i$ base for the polynomial algebra.

Now write naturally,

$$d\theta^i + \frac{1}{2} c^i_J k \theta^J \theta^K = u^i$$

So S^* should be graded ($\deg u^i = 2$)
Now get formulas for u^i_p

$$du^i + c^i_{Jk} \theta^J u_k = 0$$

(hidden within is $\partial Z = 0$)... This is a theorem of Andre Weil & Co. & Chern. In any case, these equations play the role of E.
Exercise:

$$H^*(W(g)) = \begin{array}{ll} \mathbb{R}; & 0 \\ 0; & \text{else} \end{array}$$

Now what plays the role of BG? i.e., what do you mean algebraically by an action/fibering?

$$\begin{array}{ll} \Omega^*(y) & \\ \uparrow & \text{Basic Subcomplex.} \\ \Omega^*(x) & \end{array}$$

You can find the base, when you have a free action by looking at certain subforms.
Example

$$w = a(x, y)dx + b(x, y)dy$$

where does this come from? projections?
$i \frac{\partial}{\partial y} w = b(x, y) = 0$
$\mathrm{L}_z = dz_z + i_z d$
Answer: $i_z w = 0$; z is any vector field in fiber dimension.
$\mathrm{L}_z w = 0$ where L_z is the Lie derivative.
So in general...
$y \in g$

$$i_y \theta^J = \theta^J(y)$$

$$z_y u = 0$$

$$\mathrm{L}y = di_y + i_y d$$

At last: H^*(Basic Subcomplex of $W(g)$)=$H^*(BG)$

Final examples:

$$S^1: \;\; W, d\theta = u$$

$$i_y\theta = 1, \; u \text{ is basic}$$

$$\Longrightarrow H^*(W) = \mathbb{R}[-u] \;\; \text{where } u \text{ in } \dim 2$$

(check with $H^*(\mathbb{C}P_n)$)
We can give the correct way to define equivariant cohomology, even if the action is not free. If you look at Cartan's paper, he uses the mixing diagram, and that is right. The homotopy theorists say that given a bad action replace it with a good one of the same homotopy type (replace it with the diagonal action).
i.e.,

$$H^*_G(M) = H^*(M \times E/G)$$

This measures how the group behaves and how it acts correctly for the quotient. So what is the DeRham version of equivariant cohomology? i.e., What is DeRham Equivariant theory?
Answer:
Take Basic Subcomplex of $\Omega(M) \otimes W(g)$.
This amounts to (for S^1) the Cartan model:

$$\mathrm{Inv}_{S^1}\Omega^*(M)[-u]$$

Define

$$D(w, u^k) = (dw - i_x w \cdots u)u^k$$

(So $Du = 0$)
x is a vector field which is the infinitesimal generator of S^1 action!

Homework

$$w = \frac{xdydz - ydxdx + zdxdy}{(x^2+y^2+z^2)^{3/2}}$$

invariant under S^1 -axiom but you'll also find that $i_x w = -zdz$.
So to get something nonzero, you have to add something to w. Try $w + zu$, which is the moment map of this action. This relates to the momentum map.

Example: Starting with a symplectic manifold and an S^1-action, you can find an $f \Longrightarrow w + fu$ which give a perfect stationary phase expansion. (This is also related to the Duistermatt formula.)

Physicists:

Began looking from gauge transformations. Related to super symmetry, in some vague infinite dimensional setting.

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