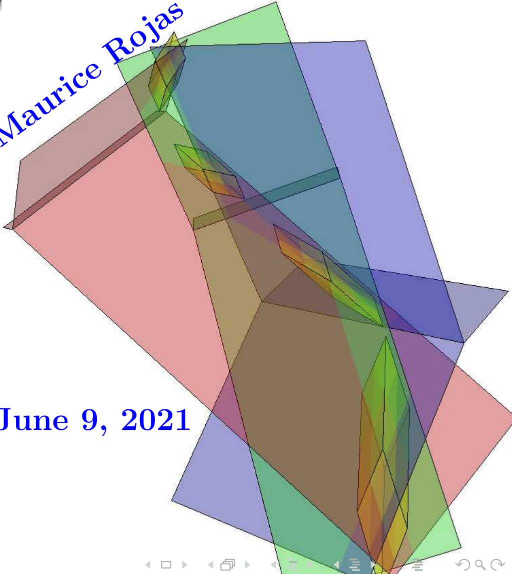
Counting Real Roots in Polynomial-Time for Systems Supported on Circuits



J. Maurice Rojas

June 9, 2021

1. Motivation & Background

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2. Fewnomials and Number Theory

3. Faster Real Root Counting for Circuit Systems

We'd like to solve...

...systems like:

$$
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2x_1^{36}x_2^{194}x_3^{50}x_4^{82}x_5^{60}+x_1^{76}x_2^{240}x_4^{41}x_5+x_1^{74}x_2^{179}x_3^{25}x_5^{57}+x_1^{25}x_2^{203}x_3^{44}x_4+x_1^{20}x_2^{167}x_3^{64}x_4^{12}x_5^{68}-37137x_1^{58}x_2^{194}x_3^{24}x_4^{36}x_5^{25}-\frac{9}{2}x_3^{166}x_4^{68}x_5^{343}, \\
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But how?

More importantly, why?

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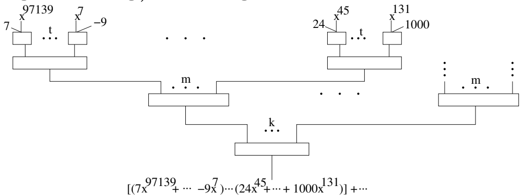

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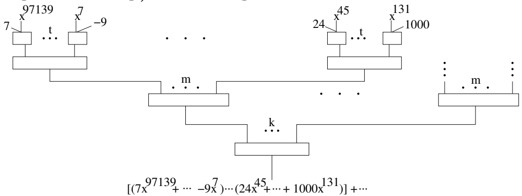

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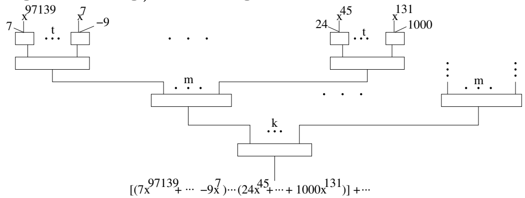

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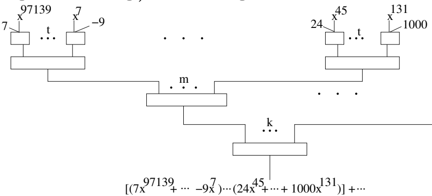

Fundamental Idea: Theorems about existence of roots are close to the **P** vs. **NP** Problem. Theorems about the deeper structure of polynomials are close to derandomization, i.e., the **P** vs. **BPP** Problem [Koiran, '11; Dutta, Saxena, Thierauf, '20].

Descartes' Rule, Biochemistry, and Learning Theory...




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But what about *exact counting*?

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- But what about systems?

Counting Roots for $(n+1)$-nomial $n \times n$ Systems is Easy, but...

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Over 245 million complex roots!

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[Bihan, Rojas, Sottile '07]: $\exists$ *systems with* $\left\lfloor \frac{n+k-1}{\min\{n,k-1\}} \right\rfloor^{\min\{n,k-1\}}$ *positive roots.*

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[Bertrand, Bihan, Sottile '06]: Tight *bound for* $k = 2$ *of* $\boxed{n+1}$.

Fewnomials and the Quest for Tight Bounds: 2010s

  

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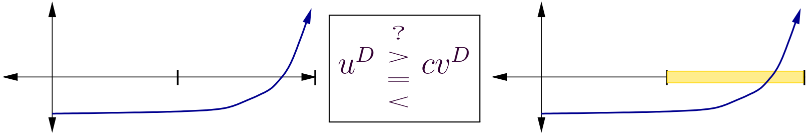

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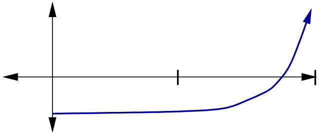
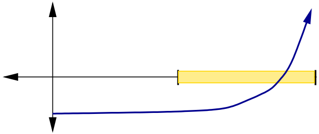


Trouble?



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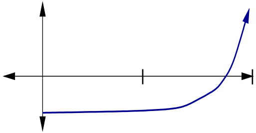

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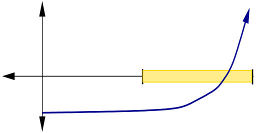

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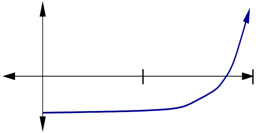

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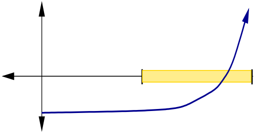

[Baker, 1966] If $a_i, b_i \in \mathbb{Z}$ with $A := \max_i \log|a_i|$, $B := \max_i \log|b_i|$, and $\Lambda := \sum_{i=1}^{n} b_i \log a_i$, then $\Lambda \neq 0 \Longrightarrow |\log|\Lambda|| = O(A)^n \log B$.

Real Univariate Trinomials?

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- So bisection is obstructed!

- Despite nice progress by [Jindal, Sagraloff, 2017] on *coarse* approximants, real root counting in time $(t \log(dH))^{O(1)}$ is still an open problem!

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•Sufficiently refined versions of the *abc-Conjecture* can reduce the complexity to polynomial in n...

Key Idea #1: Gaussian Elimination

By Gauss-Jordan Elimination applied to the linear combinations of monomials, our original 5×5 example can be reduced to

$$\begin{aligned}
x_1^{36}x_2^{194}x_3^{-116}x_4^{14}x_5^{-283} &= 16384cx_1^{58}x_2^{194}x_3^{-142}x_4^{-32}x_5^{-318} + \frac{1}{4} \\
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Next: Observe that the set of exponent vectors is a *circuit in the sense of combinatorics*: It is the union of a point and the vertex set of a simplex...

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provided some additional sign conditions are met...

Note: The exponents are usually *much* larger!

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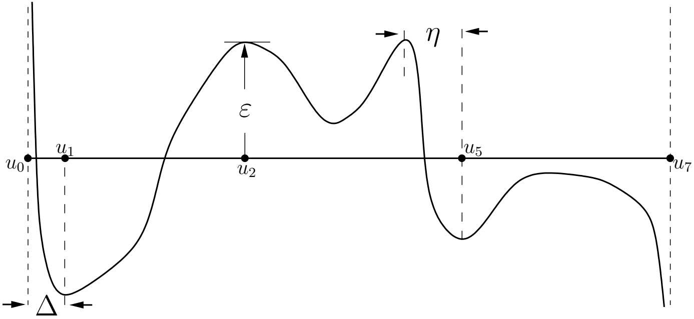

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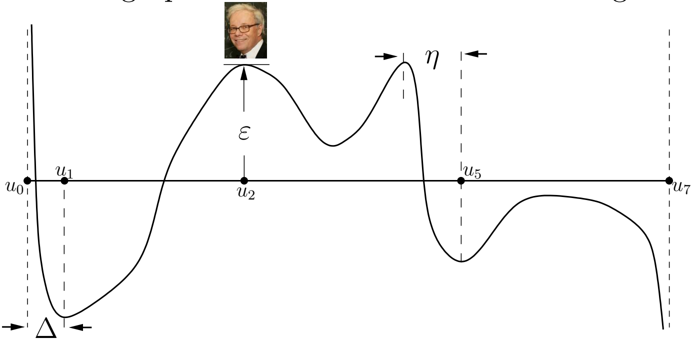

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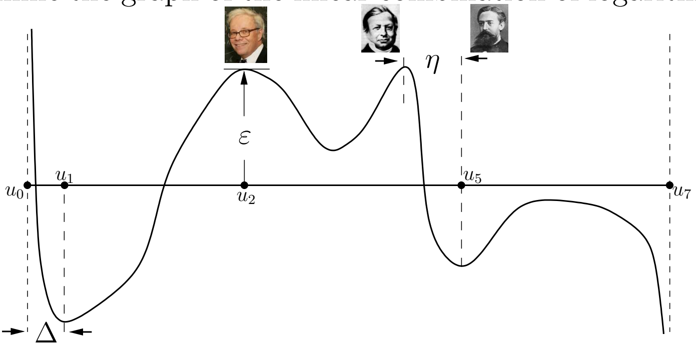

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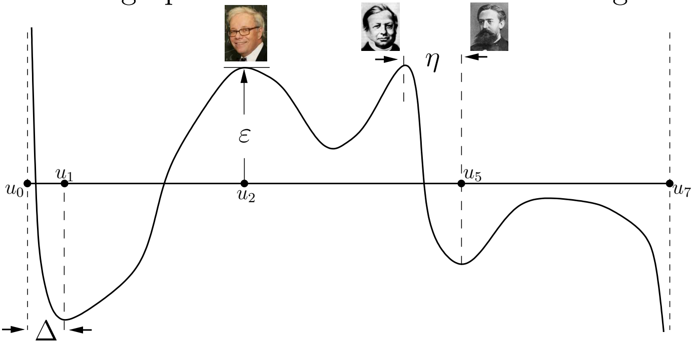

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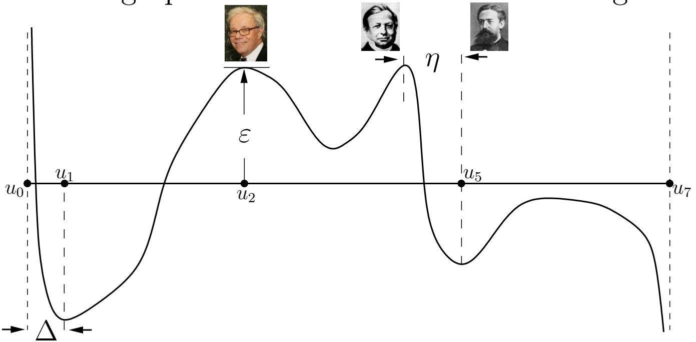

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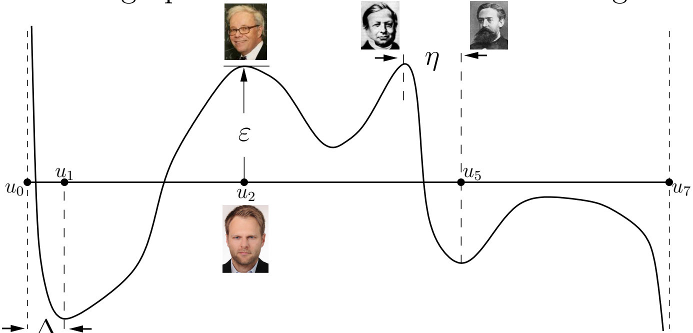

Baker-*Wustholtz* gives height of peaks... Bounds of Liouville and Markov control Δ and η... Then use

Rolle's Theorem, AGM Iteration for accurate logs, and Sturm-Habicht sequences to isolate critical points!...

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While we can now count real roots in time $(\log(dH))^{O(n)}$, there is growing evidence that we can attain complexity $(n\log(DH))^{O(1)}$ *on average* [Deng, Ergür, Paouris, Rojas, 2021]...

♡ Thank you for your attention!

See `www.math.tamu.edu/~rojas` for preprints and further info...