# NEW SUBEXPONENTIAL FEWNOMIAL HYPERSURFACE BOUNDS

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Abstract. Suppose $c_1,\ldots,c_{n+k}$ are real numbers, $\{a_1,\ldots,a_{n+k}\}\subset\mathbb{R}^n$ is a set of points not all lying in the same affine hyperplane, $y\in\mathbb{R}^n$, $a_j\cdot y$ denotes the standard real inner product of $a_j$ and $y$, and we set $g(y):=\sum_{j=1}^{n+k}c_je^{a_j\cdot y}$. We prove that, for generic $c_j$, the number of connected components of the real zero set of $g$ is $O\left(n^2+\sqrt{2}^{k^2}(n+2)^{k-2}\right)$. The best previous upper bounds, when restricted to the special case $k=3$ and counting just the non-compact components, were already exponential in $n$.

## 1. Introduction

Estimating the number of connected components of the real zero set of a system of polynomial equations is a fundamental problem occuring in numerous applications. For instance, in robotics [WMS92, CM93], chemical reaction networks [JS17], economic modelling [McL05], and complexity theory [Koi11], information on the topology of the underlying zero set is sometimes at least as important as numerically approximating solutions. We derive topological bounds in the broader context of real exponential sums, significantly sharpening older bounds from fewnomial theory [Kho91, BS09].

**Definition 1.1.** *For any field $K$ we let $K^*:=K\setminus\{0\}$. Let $\mathcal{A}\in\mathbb{R}^{n\times(n+k)}$ have $j^{\underline{th}}$ column $a_j$ and let $c_1,\ldots,c_{n+k}\in\mathbb{R}^*$. We then call $g(y):=\sum_{j=1}^{t}c_je^{a_j\cdot y}$ a* (real) $n$-variate exponential $(n+k)$-sum, *and call $\mathcal{A}$ the* spectrum *of $g$. We also let $c_g:=(c_1,\ldots,c_{n+k})$. Finally, for any function $h:\mathbb{C}^n\longrightarrow\mathbb{R}$, we let $Z_{\mathbb{C}}(h)$, $Z_{\mathbb{R}}(h)$, and $Z_+(h)$ respectively denote the zeroes of $h$ in $\mathbb{C}^n$, $\mathbb{R}^n$, and $\mathbb{R}^n_+$ (the positive orthant).* $\diamond$

Note that when $\mathcal{A}\in\mathbb{Z}^{n\times(n+k)}$ there is an obvious $f\in\mathbb{R}\left[x_1^{\pm1},\ldots,x_n^{\pm1}\right]$, with exactly $n+k$ monomial terms, such that $g(y)=f(e^{y_1},\ldots,e^{y_n})$ identically, and the zero sets $Z_{\mathbb{R}}(g)$ and $Z_+(f)$ have the same number of connected components. In this sense, among many others, real exponential sums generalize real polynomials.

We say a condition involving a tuple of real parameters $(z_1,\ldots,z_N)$ holds *generically* if and only if the set of choices of $(z_1,\ldots,z_N)$ making the condition true is dense and open in $\mathbb{R}^N$. For instance, it is easy to show that for generic $\mathcal{A}\in\mathbb{R}^{n\times(n+k)}$ (with $k\geq1$) we have that $\{a_1,\ldots,a_{n+k}\}$ do not all lie in the same affine hyperplane.

**Theorem 1.2.** *Suppose $g$ is an $n$-variate $(n+k)$-sum with spectrum $\mathcal{A}$ and $\{a_1,\ldots,a_{n+k}\}$ do not all lie in the same affine hyperplane. Then, for generic $c_g$, $Z_{\mathbb{R}}(g)$ has no more than $\frac{(n+k)(n+k-1)}{2}+\left\lfloor\frac{e^2+3}{4}\sqrt{2}^{(k-2)(k-3)}(n+2)^{k-2}\right\rfloor$ connected components. Furthermore, for $k=3$, a sharper upper bound of $\frac{(n+3)(n+2)}{2}+\left\lfloor\frac{n+5}{2}\right\rfloor$ holds.*

We prove Theorem 1.2 in Section 3.1 below. The best previous upper bound on the number of connected components, [BS09, Thm. 1], came from a larger topological invariant: the sum of the *Betti numbers* of the underlying zero set. (See also [Bas99] for an important precursor in the semi-algebraic setting.) Our bound is polynomial in $n$ for any fixed $k$, while the bound from [BS09, Thm. 1] is exponential in each of $n$ and $k$. For $k\in\{1,2\}$ respective optimal upper bounds of 1 and 2 are already known (see, e.g., [BRS09, Bih11, BPRRR17]).

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## 2. Background

A central tool behind the proof of Theorem 1.2 is an extension of Gelfand, Kapranov, and Zelevinsky's theory of $\mathcal{A}$-discriminants [GKZ94] to exponential sums. This generalization was first developed in [RR17].

**Definition 2.1.** *For any* $\mathcal{A}\in\mathbb{R}^{n\times(n+k)}$ *we define the* generalized $\mathcal{A}$-discriminant variety, $\Xi_{\mathcal{A}}$, *to be the Euclidean closure of the set of all* $[c_1 : \cdots : c_{n+k}]\in\mathbb{P}^{n+k-1}_{\mathbb{C}}$ *such that* $\sum_{j=1}^{n+k} c_j e^{a_j\cdot z}$ *has a degenerate root in* $\mathbb{C}^n$. *Also, we call* $\mathcal{A}$ non-defective *if and only if* $\Xi_{\mathcal{A}}$ *has codimension* 1 *in* $\mathbb{P}^{n+k-1}_{\mathbb{C}}$. $\diamond$

**Definition 2.2.** *Given any two subsets* $X, Y\subseteq\mathbb{R}^n$, *an* isotopy from $X$ to $Y$ (ambient in $\mathbb{R}^n$) *is a continuous map* $I : [0,1]\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ *satisfying (1)* $I(t,\cdot)$ *is a homeomorphism for all* $t\in[0,1]$, *(2)* $I(0,x)=x$ *for all* $x\in\mathbb{R}^n$, *and (3)* $I(1,X)=Y$. $\diamond$

It is easily checked that an isotopy from $X$ to $Y$ implies an isotopy from $Y$ to $X$ as well. So isotopy is in fact an equivalence relation and it makes sense to speak of *isotopy type*.

The real part of $\Xi_{\mathcal{A}}$ (along with some additional pieces: see Theorems 3.1 and 3.6 below) partitions the coefficient space of $g$ into regions where $Z_{\mathbb{R}}(g)$ is smooth and the isotopy type of $Z_{\mathbb{R}}(g)$ is constant. Moreover, since scaling variables and coefficient vectors does not affect the presence of singularities in $Z_{\mathbb{R}}(g)$, the variety $\Xi_{\mathcal{A}}$ has certain homogeneities. As we'll see below, these homogeneities can be quotiented out to better study regions of the coefficient space where $Z_{\mathbb{R}}(g)$ is smooth and has constant isotopy type. For any $S\subseteq\mathbb{C}^N$ we let $\overline{S}$ denote the Euclidean closure of $S$.

**Definition 2.3.** *For any* $\mathcal{A}\in\mathbb{R}^{n\times(n+k)}$ *let* $\widehat{\mathcal{A}}\in\mathbb{R}^{(n+1)\times(n+k)}$ *denote the matrix with first row* $[1,\ldots,1]$ *and bottom* $n$ *rows forming* $\mathcal{A}$, *and set* $d(\mathcal{A}):=\mathrm{Rank}\widehat{\mathcal{A}}-1$. *Let* $B\in\mathbb{R}^{(n+k)\times(n+k-d(\mathcal{A})-1)}$ *be any matrix whose columns form a basis for the right nullspace of* $\widehat{\mathcal{A}}$. *Let* $\beta_i$ *denote the* $i^{\underline{th}}$ *row of* $B$, *let* $(\cdot)^\top$ *denote matrix transpose, and for any* $z=(z_1,\ldots,z_N)$ *let* $\mathrm{Log}|z| := (\log|z_1|,\ldots,\log|z_N|)$. *When* $\mathcal{A}$ *is non-defective we then set* $\lambda:=(\lambda_1,\ldots,\lambda_{n+k-d(\mathcal{A})-1})$ *and define the (projective) hyperplane arrangement*

$$H_{\mathcal{A}}:=\{[\lambda] \mid \lambda\cdot\beta_i=0 \text{ for some (nonzero) row } \beta_i \text{ of } B\}\subset\mathbb{P}^{n+k-d(\mathcal{A})-2}_{\mathbb{C}}.$$

*Finally, we define* $\xi_{\mathcal{A},B} : \left(\mathbb{P}^{n+k-d(\mathcal{A})-2}_{\mathbb{C}}\setminus H_{\mathcal{A}}\right) \longrightarrow \mathbb{R}^{n+k-d(\mathcal{A})-1}$ *by* $\xi_{\mathcal{A},B}([\lambda]):=\left(\mathrm{Log}\left|\lambda B^\top\right|\right)B$. *(So* $\xi_{\mathcal{A},B}$ *is defined by multiplying a row vector by a matrix.) We then call* $\Gamma(\mathcal{A},B) := \overline{\xi_{\mathcal{A},B}\left(\mathbb{P}^{n+k-d(\mathcal{A})-2}_{\mathbb{R}}\setminus H_{\mathcal{A}}\right)}$ a reduced discriminant contour. $\diamond$

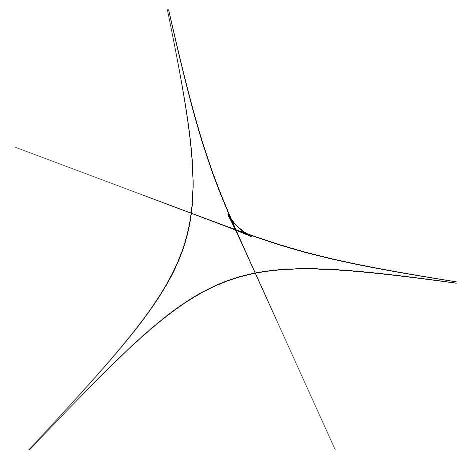
For any subset $S\subseteq\mathbb{R}^n$, we let $\mathrm{Conv}S$ denote the smallest convex set containing $S$. It is easily checked that $\dim\mathrm{Conv}\{a_1,\ldots,a_{n+k}\}=d(\mathcal{A})$ and thus, for generic $\mathcal{A}$, we have $d(\mathcal{A})=n$. However, we will need to consider arbitrary $d(\mathcal{A})$ in order to more easily describe our approach to counting isotopy types. Let us call $\mathcal{A}$ *pyramidal* if and only if $\mathcal{A}$ has a column $a_j$ such that $\{a_1,\ldots,a_{j-1},a_{j+1},\ldots,a_{n+k}\}$ lies in a $(d(\mathcal{A})-1)$-dimensional affine subspace. The following proposition, on certain exceptional spectra $\mathcal{A}$, will prove useful later on.

**Proposition 2.4.** *Following the preceding notation,* $\mathcal{A}$ *is pyramidal if and only if* $B$ *has a zero row. In particular,* $\mathcal{A}$ *non-defective implies that* $\mathcal{A}$ *is not pyramidal.* ■

**Remark 2.5.** *When* $\mathcal{A}\in\mathbb{Z}^{n\times(n+k)}$ *and* $\mathcal{A}$ *is non-defective it follows easily from the development of* [RR17] *that* $\overline{\xi_{\mathcal{A},B}\left(\mathbb{P}^{n+k-d(\mathcal{A})-2}_{\mathbb{C}}\setminus H_{\mathcal{A}}\right)}$ *is in fact a linear section of the* amoeba *of the classical* $\mathcal{A}$-discriminant polynomial $\Delta_{\mathcal{A}}$. $\xi_{\mathcal{A},B}$ *is thus a generalization of the (logarithmic)* Horn-Kapranov Uniformization *(see* [Kap91, GKZ94]*). See also* [PT05] *for further background on* $\mathcal{A}$*-discriminant contours in the special case* $\mathcal{A}\in\mathbb{Z}^{n\times(n+k)}$. $\diamond$

**Theorem 2.6.** [RR17, Thm. 1.7] *If* $\mathcal{A}$ *is non-defective then* $\Gamma(\mathcal{A},B)$ *is a finite union of codimension* 1 *smooth semi-analytic subsets of* $\mathbb{R}^{n+k-d(\mathcal{A})-1}$. *Furthermore, there is a codimension-*2 *semi-analytic set* $Y\subset\mathbb{R}^{n+k-d(\mathcal{A})-1}$ *such that* $\Gamma(\mathcal{A},B)\cup Y=\left(\mathrm{Log}\left|\Xi_{\mathcal{A}}\cap\mathbb{P}^{n+k-1}_{\mathbb{R}}\right|\right)B$. ■

**Example 2.7.** *When* $\mathcal{A} := \begin{bmatrix} 0 & 1 & 0 & 4 & 1 \\ 0 & 0 & 1 & 1 & 4 \end{bmatrix}$ *we are in essence considering the family of exponential sums* $g(y) := f(e^{y_1}, e^{y_2})$ *where* $f(x) = c_1 + c_2x_1 + c_3x_2 + c_4x_1^4x_2 + c_5x_1x_2^4$. *A suitable* $B$ *(among many others) with columns defining a basis for the right nullspace of* $\widehat{\mathcal{A}}$ *is then* $B \approx \begin{bmatrix} 0.5079 & -0.8069 & 0.1721 & 0.2267 & -0.0997 \\ 0.5420 & 0.1199 & -0.7974 & -0.0851 & 0.2206 \end{bmatrix}^{\top}$, *and the corresponding reduced contour* $\Gamma(\mathcal{A},B)$, *intersected with* $[-4,4]$, *is drawn to the right.* $\diamond$



In what follows, we set

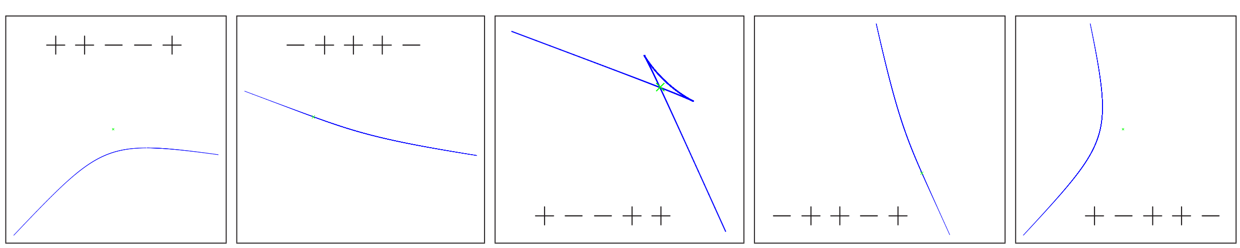
$$\mathrm{sign}(c_g) := (\mathrm{sign}(c_1), \ldots, \mathrm{sign}(c_{n+k})) \in \{\pm 1\}^{n+k}.$$

**Definition 2.8.** *Suppose* $\mathcal{A}\in\mathbb{R}^{n\times(n+k)}$ *is non-defective and* $\sigma=(\sigma_1,\ldots,\sigma_{n+k})\in\{\pm1\}^{n+k}$. *We then call*

$$\Gamma_\sigma(\mathcal{A},B) := \overline{\left\{\xi_{\mathcal{A},B}([\lambda]) \mid \mathrm{sign}\,(\lambda B^\top) = \pm\sigma\ ,\ [\lambda]\in\mathbb{P}^{n+k-d(\mathcal{A})-2}_{\mathbb{R}}\setminus H_{\mathcal{A}}\right\}} \subset \mathbb{R}^{n+k-d(\mathcal{A})-1}$$

*a* signed reduced contour, *and we call any connected component* $\mathcal{C}$ *of* $\mathbb{R}^{n+k-d(\mathcal{A})-1}\setminus\Gamma_\sigma(\mathcal{A},B)$ *a* reduced signed chamber. *We also call* $\mathcal{C}$ *an* outer *or* inner chamber, *according as* $\mathcal{C}$ *is unbounded or bounded.* $\diamond$

**Example 2.9.** *Continuing Example 2.7, there are* 16 *possible choices for* $\sigma$, *if we identify sign sequences with their negatives. Among these choices, there are* 11 $\sigma$ *yielding* $\Gamma_\sigma(\mathcal{A},B)=\emptyset$. *The remaining choices, along with their respective* $\Gamma_\sigma(\mathcal{A},B)$ *are drawn below.* $\diamond$




Note that the curves drawn above are in fact unbounded, so the number of reduced signed chambers for the $\sigma$ above, from left to right, is respectively 2, 2, 3, 2, and 2. (The tiny $\times$ in each illustration indicates the origin in $\mathbb{R}^2$.) In particular, only $\sigma=(1,-1,-1,1,1)$ yields an inner chamber. Note also that $\Gamma(\mathcal{A},B)$ is always the union of all the $\Gamma_\sigma(\mathcal{A},B)$.

**Remark 2.10.** *While the shape of the reduced signed chambers certainly depends on the choice of* $B$, *the hyperplane arrangement* $H_{\mathcal{A}}$ *and the number of signed chambers for any fixed* $\sigma$ *are independent of* $B$. *In particular, working with the* $\Gamma_\sigma(\mathcal{A},B)$ *in* $\mathbb{R}^{n+k-d(\mathcal{A})-1}$ *helps us visualize and work with* $\Xi_{\mathcal{A}}$, *which lives in* $\mathbb{P}^{n+k-1}_{\mathbb{R}}$. $\diamond$

## 3. Morse Theory, Fewnomial Bounds, and the Proof of Theorem 1.2

Let us call $\mathcal{A}\in\mathbb{R}^{n\times(n+k)}$ *combinatorially simplicial* if and only if $\mathcal{A}\cap Q$ has cardinality $1+\dim Q$ for every face $Q$ of $\mathrm{Conv}\{a_1,\ldots,a_{n+k}\}$. (The books [Grü03, Zie95] are excellent standard references on polytopes, their faces, and their normal vectors.) Note that $\mathrm{Conv}\{a_1,\ldots,a_{n+k}\}$ need *not* be a simplex for $\mathcal{A}$ to be combinatorially simplicial (consider, e.g., Example 2.7). We now state the main reason we care about reduced signed chambers.

**Theorem 3.1.** [RR17, Thm. 3.1] *Suppose $\mathcal{A}\in\mathbb{R}^{n\times(n+k)}$ is combinatorially simplicial, non-defective, and $g_1$ and $g_2$ are each $n$-variate exponential $(n+k)$-sums with spectrum $\mathcal{A}$. Suppose further that* $\mathrm{sign}(c_{g_1})=\pm\mathrm{sign}(c_{g_2})$*, and* $(\mathrm{Log}|c_{g_1}|)B$ *and* $(\mathrm{Log}|c_{g_2}|)B$ *lie in the same reduced discriminant chamber. Then $Z_{\mathbb{R}}(g_1)$ and $Z_{\mathbb{R}}(g_2)$ are ambiently isotopic in $\mathbb{R}^n$.* ■

The special case $\mathcal{A}\in\mathbb{Z}^{n\times(n+k)}$, without the use of Log or $B$, is alluded to near the beginning of [GKZ94, Ch. 11, Sec. 5]. However, Theorem 3.1 is really just an instance of *Morse Theory* [Mil69, GM88], once one considers the manifolds defined by the fibers of the map $Z_{\mathbb{R}}(g)\mapsto(\mathrm{Log}|c_g|)B$ along paths inside a fixed signed chamber. In particular, the assumption that $\mathcal{A}$ be combinatorially simplicial forces any topological change in $Z_{\mathbb{R}}(g)$ to arise solely from singularities of $Z_{\mathbb{R}}(g)$ in $\mathbb{R}^n$. When $\mathcal{A}$ is more general, topological changes in $Z_{\mathbb{R}}(g)$ can arise from pieces of $Z_{\mathbb{R}}(g)$ approaching infinity, with no singularity appearing in $\mathbb{R}^n$. So our chambers will need to be cut into smaller pieces.

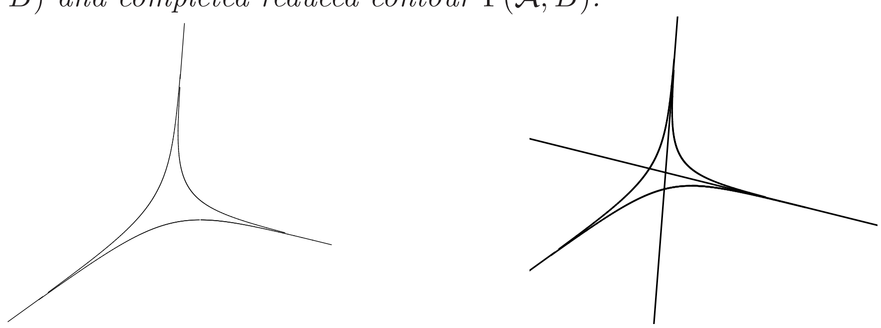
So we now address arbitrary $\mathcal{A}$, but we'll first need a little more terminology.

**Definition 3.2.** *Given any $\mathcal{A}\in\mathbb{R}^{n\times(n+k)}$ with distinct columns, and any outer normal $w\in\mathbb{R}^n$ to a face of* $\mathrm{Conv}\mathcal{A}$*, we let $\mathcal{A}^w:=[a_{j_1},\ldots,a_{j_r}]$ denote the sub-matrix of $\mathcal{A}$ corresponding to the set $\{a\in\mathcal{A}\mid a\cdot w=\max_{a'\in\mathcal{A}}\{a'\cdot w\}\}$. We call $\mathcal{A}^w$ a* (proper) non-simplicial face *of $\mathcal{A}$ when $d(\mathcal{A}^w)\leq d(\mathcal{A})-1$ and $\mathcal{A}^w$ has at least $d(\mathcal{A}^w)+1$ columns. Also let $B^w$ be any matrix whose columns form a basis for the right nullspace of $\widehat{(\mathcal{A}^w)}$, and let $\pi_w:\mathbb{C}^{n+k}\longrightarrow\mathbb{C}^r$ be the natural coordinate projection map defined by $\pi_w(c_1,\ldots,c_{n+k}):=(c_{j_1},\ldots,c_{j_r})$. When $\mathcal{A}$ is non-defective and* not *combinatorially simplicial we then define the* completed *reduced signed contour, $\widetilde{\Gamma}_\sigma(\mathcal{A},B)\subset\mathbb{R}^{n+k-d(\mathcal{A})-1}$, to be the union of $\overline{\Gamma_\sigma(\mathcal{A},B)}$ and*

$$\bigcup_{\substack{\mathcal{A}^w \text{ a non-}\\ \text{simplicial}\\ \text{face of } \mathcal{A}}}\overline{\left\{\pi_w^{-1}(\mathrm{Log}|\lambda(B^w)^\top|)\,B \;\middle|\; \mathrm{sign}\,(\lambda(B^w)^\top)=\pm\pi_w(\sigma)\ ,\ [\lambda]\in\mathbb{P}_{\mathbb{R}}^{n+k-d(\mathcal{A})-2}\setminus H_{\mathcal{A}}\right\}}.$$

*We call any unbounded connected component of $\mathbb{R}^{n+k-d(\mathcal{A})-1}\setminus\widetilde{\Gamma}_\sigma(\mathcal{A},B)$ an* outer chamber*. Finally, we define $\widetilde{\Gamma}(\mathcal{A},B):=\bigcup_{\sigma\in\{\pm1\}^{n+k}}\widetilde{\Gamma}_\sigma(\mathcal{A},B)$.* ⋄

**Example 3.3.** *When $\mathcal{A}=\begin{bmatrix}0&1&0&2&0\\0&0&1&0&2\end{bmatrix}$ it is easy to find a $B$ yielding the following reduced contour $\Gamma(\mathcal{A},B)$ and completed reduced contour $\widetilde{\Gamma}(\mathcal{A},B)$:*



*Note in particular that* $\widetilde{\Gamma}(\mathcal{A},B)=\Gamma(\mathcal{A},B)\cup S_1\cup S_2$ *where* $S_1$ *and* $S_2$ *are lines that can be viewed as line bundles over points. These points are in fact* $(\mathrm{Log}|\Xi_{\mathcal{A}_1}|)B$ *and* $(\mathrm{Log}|\Xi_{\mathcal{A}_2}|)B$ *where* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *are the facets of* $\mathcal{A}$ *with respective outer normals* $(-1,0)$ *and* $(0,-1)$*, and* $B\approx\begin{bmatrix}0.4335 & -0.8035 & -0.0635 & 0.4018 & 0.0317\\ 0.3127 & 0.2002 & -0.8256 & -0.1001 & 0.4128\end{bmatrix}^\top$. $\diamond$

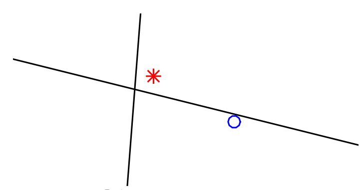
**Proposition 3.4.** *If* $\mathcal{A}$ *is not combinatorially simplicial then* $\mathcal{A}$ *has at most* $n+k-d(\mathcal{A})-1$ *non-simplicial faces.* ■

**Proposition 3.5.** *Suppose* $k=3$*,* $\mathcal{A}$ *has exactly* 2 *non-simplicial facets,* $d(\mathcal{A})=n$*,* $B\in\mathbb{R}^{(n+3)\times 2}$ *is any matrix whose columns form a basis for the right nullspace of* $\widehat{\mathcal{A}}$*, and* $[\beta_{i,1},\beta_{i,2}]$ *is the* $i^{\underline{th}}$ *row of* $B$*. Then* $\{[\beta_{i,1}:\beta_{i,2}]\}_{i\in\{1,\ldots,n+3\}}$ *has cardinality* $n+1$ *as a subset of* $\mathbb{P}^1_{\mathbb{R}}$*, and* $\widetilde{\Gamma}(\mathcal{A},B)\setminus\Gamma(\mathcal{A},B)$ *is a union of* 2 *lines.* ■

**Theorem 3.6.** [RR17, Thm. 3.8] *Suppose* $\mathcal{A}\in\mathbb{R}^{n\times(n+k)}$ *is non-defective,* not *combinatorially simplicial, and* $d(\mathcal{A})=n$*. Suppose also that* $g_1$ *and* $g_2$ *are each* $n$*-variate exponential* $(n+k)$*-sums with spectrum* $\mathcal{A}$*,* $\sigma:=\mathrm{sign}(c_{g_1})=\pm\mathrm{sign}(c_{g_2})$*, and* $(\mathrm{Log}|c_{g_1}|)B$ *and* $(\mathrm{Log}|c_{g_2}|)B$ *lie in the same connected component of* $\mathbb{R}^{n+k-d(\mathcal{A})-1}\setminus\widetilde{\Gamma}_\sigma(\mathcal{A},B)$*. Then* $Z_{\mathbb{R}}(g_1)$ *and* $Z_{\mathbb{R}}(g_2)$ *are ambiently isotopic in* $\mathbb{R}^n$*.* ■

**Example 3.7.** *Observe that the circle defined by* $\left(u+\frac{1}{2}\right)^2+(v-2)^2=1$ *intersects the positive orthant, while the circle defined by* $\left(u+\frac{3}{2}\right)^2+\left(v-\frac{3}{2}\right)^2=1$ *does not. Consider then* $\mathcal{A}=\begin{bmatrix}0&1&0&2&0\\0&0&1&0&2\end{bmatrix}$ *as in our last example, and let* $g_1=\left(e^{y_1}+\frac{1}{2}\right)^2+\left(e^{y_2}-2\right)^2-1$ *and* $g_2=\left(e^{y_1}+\frac{3}{2}\right)^2+\left(e^{y_2}-\frac{3}{2}\right)^2-1$*. Then* $g_1$ *and* $g_2$ *have spectrum* $\mathcal{A}$*,* $\mathrm{sign}(g_1)=\mathrm{sign}(g_2)=\sigma$ *with* $\sigma=(1,1,-1,1,1)$*, and* $(\mathrm{Log}|c_{g_1}|)B$ *and* $(\mathrm{Log}|c_{g_2}|)B$ *lie in the same reduced signed* $\mathcal{A}$*-discriminant chamber (since* $\Gamma_\sigma(\mathcal{A},B)=\emptyset$ *here). However,* $Z_{\mathbb{R}}(g_1)$ *consists of a single smooth arc, while* $Z_{\mathbb{R}}(g_2)$ *is empty. This is easily explained by the* completed *contour* $\widetilde{\Gamma}_\sigma(\mathcal{A},B)$ *consisting of two lines, and* $(\mathrm{Log}|c_{g_1}|)B$ *and* $(\mathrm{Log}|c_{g_2}|)B$ *lying in distinct connected components of* $\mathbb{R}^2\setminus\widetilde{\Gamma}_\sigma(\mathcal{A},B)$ *as shown, respectively via the symbols* $\circ$ *and* $*$*, below to the right.* $\diamond$

Although we defined signed contours via a transcendental parametrization, they obey certain tameness properties akin to algebraic sets. One fundamental result implying this tameness is the following refined fewnomial bound.



**Theorem 3.8.** *(See* [BS07, Thm. 3.1], [BBS05], *&* [PR13, Lem. 1.8]*) Suppose* $m\geq 1$*,* $j\geq 2$*,* $E=[e_{i,\ell}]\in\mathbb{R}^{(m+j)\times j}$*,* $U:=[u_{i,\ell}]\in\mathbb{R}^{(m+j)\times(j+1)}$ *has* $i^{\underline{th}}$ *row* $(u_{i,0},u_{i,1},\ldots,u_{i,j})$*,* $u_i:=(u_{i,1},\ldots,u_{i,j})$*,* $\Delta:=\{y\in\mathbb{R}^j \mid u_{i,0}+u_i\cdot y>0$ *for all* $i\in\{1,\ldots,j\}\}$*, and*

$$H:=\left(\prod_{\ell=1}^{m+j}(u_{\ell,0}+u_\ell\cdot y)^{e_{\ell,1}},\ldots,\prod_{\ell=1}^{m+j}(u_{\ell,0}+u_\ell\cdot y)^{e_{\ell,j}}\right)-(1,\ldots,1).$$

*Then* $H$ *has fewer than* $S(m,j):=\frac{e^2+3}{4}\left(\sqrt{2^{j-1}}m\right)^j$ *non-degenerate roots in* $\Delta$*. Furthermore, for* $j=1$*,* $H$ *has at most* $S(m,1):=m+1$ *non-degenerate roots in* $\Delta$*, and there exist* $H$ *attaining* $m+1$ *distinct roots in* $\Delta$*.* ■

We call systems of the above form *$j$-variate Gale Dual systems with $m+j$ factors.*

**Corollary 3.9.** *Suppose* $\mathcal{A}\in\mathbb{R}^{n\times(n+k)}$ *is combinatorially simplicial, non-defective,* $d(\mathcal{A})=n$*, and* $\sigma\in\{\pm1\}^{n+k}$*. Then, following the notation of Theorem 3.8, a generic affine line* $L\subset$

$\mathbb{R}^{k-1}$ *intersects* $\Gamma_\sigma(\mathcal{A},B)$ *in no more than* $\lfloor S(n+2,k-2)\rfloor$ *points when* $k\geq 3$*. Also, for* $k=2$ *there is at most* $S(n+2,0):=1$ *intersection.*

**Proof of Corollary 3.9:** When $k=2$ we have that $\Gamma(\mathcal{A},B)$ is merely a point, so this case follows easily. So let us assume $k\geq 3$ and let $[\mathcal{L}_{i,j}]_{(i,j)\in\{1,\ldots,k-2\}\times\{0,\ldots,k-1\}}\in\mathbb{R}^{(k-2)\times k}$ be any matrix defining the affine line $L$ as follows:

$$L=\left\{x\in\mathbb{R}^{k-1}\mid \mathcal{L}_{i,1}x_1+\cdots+\mathcal{L}_{i,k-1}x_{k-1}=\mathcal{L}_{i,0} \text{ for all } i\in\{1,\ldots,k-2\}\right\}.$$

Also let $(\xi_1,\ldots,\xi_{k-1}):=\xi_{\mathcal{A},B}$. (So each $\xi_i$ is a logarithm of the absolute value of a linear form in $\lambda_1,\ldots,\lambda_{k-1}$.) Note then that $L$ meets $\Gamma(\mathcal{A},B)$ at the point $\xi_{\mathcal{A},B}([\lambda])$ only if

$$\begin{aligned}\sum_{\ell=1}^{k-1}\mathcal{L}_{1,\ell}\xi_\ell([\lambda]) &= \mathcal{L}_{1,0}\\ &\vdots\\ \sum_{\ell=1}^{k-1}\mathcal{L}_{k-2,\ell}\xi_\ell([\lambda]) &= \mathcal{L}_{k-2,0}\end{aligned}$$

Exponentiating both sides of the preceding system, and collecting factors, we obtain that there is a matrix $E=[E_{i,j}]\in\mathbb{R}^{(k-2)\times(n+k)}$ such that $L$ meets $\Gamma(\mathcal{A},B)$ at the point $\xi_{\mathcal{A},B}([\lambda])$ only if

$$\begin{aligned}\prod_{\ell=1}^{n+k}(\beta_\ell\cdot\lambda)^{E_{1,\ell}} &= e^{\mathcal{L}_{1,0}}\\ &\vdots\\ \prod_{\ell=1}^{n+k}(\beta_\ell\cdot\lambda)^{E_{k-2,\ell}} &= e^{\mathcal{L}_{k-2,0}}\end{aligned}$$

Setting $\lambda_{k-1}=1$ to dehomogenize the linear forms $\beta_i\cdot\lambda$, Theorem 3.8 then tells us that $L$ meets $\Gamma(\mathcal{A},B)$ at no more than $S(n+2,k-2)$ points. Since the number of intersections is an integer, we can take floor and conclude. ■

**Lemma 3.10.** *If* $n,k',k''\geq 2$ *then* $S(n+1,k'+k''-2)\geq S(n+1,k'-2)+S(n+1,k''-2)$*. More generally, if* $k_1+\cdots+k_r=k-1$ *with* $k_i\geq 2$ *for all* $i$ *and* $r\geq 2$*, then* $S(n+1,k-5)+1\geq \sum_{i=1}^r S(n+1,k_i-2)$.

**Proof of Lemma 3.10:** The first assertion is immediate since $S(n+1,k'-2)+S(n+1,k''-2)$ $\leq 2S(n+1,k''-2)$ (assuming $k''\geq k'$) and $2^{1+(k''-2)(k''-3)/2}\leq 2^{(k'+k''-2)(k'+k''-3)/2}$. The second assertion follows easily by induction: Writing $k=(\cdots((k_1+k_2)+k_3)+\cdots+k_{r-1})+k_r$, the first assertion of our lemma implies that $\sum_{i=1}^r S(n+1,k_i-2)\leq S(n+1,k'-2)+S(n+1,k''-2)$ for some $k',k''\geq 2$ with $k-1=k'+k''$. It is then easy to see (from the power of 2 factor of $S(m,j)$ again) that $S(n+1,k'-2)+S(n+1,k''-2)\leq S(n+1,k-3-2)+S(n+1,2-2)$, i.e., the left-hand side of the inequality is maximized when $\{k',k''\}=\{2,k-3\}$. ■

**Corollary 3.11.** *Suppose* $\mathcal{A}\in\mathbb{R}^{n\times(n+k)}$ *is non-defective,* $\mathcal{A}$ *is* not *combinatorially simplicial,* $d(\mathcal{A})=n$*, and* $\sigma\in\{\pm 1\}^{n+k}$*. Then a generic affine line* $L\subset\mathbb{R}^{k-1}$ *intersects* $\widetilde{\Gamma}_\sigma(\mathcal{A},B)$ *in no more than*

$$S(n+2,k-2)+S(n+1,k-5)+\cdots+S\left(n+2-\min\left\{n+1,\left\lfloor\tfrac{k-2}{3}\right\rfloor\right\},k-2-3\min\left\{n+1,\left\lfloor\tfrac{k-2}{3}\right\rfloor\right\}\right)$$
$$+\min\left\{n+1,\left\lfloor\tfrac{k-2}{3}\right\rfloor\right\}$$

*points when* $k\geq 4$*. Also, for* $k\in\{2,3\}$ *we have respective upper bounds of* $1$ *and* $n+5$*.*

**Proof of Corollary 3.11:** We simply follow essentially the same argument as the proof of Corollary 3.9, save that we work with $\widetilde{\Gamma}_\sigma(\mathcal{A},B)$ instead of $\Gamma_\sigma(\mathcal{A},B)$. In particular, the case $k=2$ presents no new difficulties since $\widetilde{\Gamma}_\sigma(\mathcal{A},B)$ is always a point. The case $k=3$ follows easily upon observing, thanks to Proposition 3.5, that $\widetilde{\Gamma}_\sigma(\mathcal{A},B)\setminus\Gamma_\sigma(\mathcal{A},B)$ is either empty, a line, or two lines.

For $k\geq 4$ we simply observe that $L$ will either intersect $\Gamma_\sigma(\mathcal{A},B)$ or some fiber closure of the form $\overline{\left\{\pi_w^{-1}(\text{Log}|\lambda(B^w)^\top|)\,B \mid \text{sign}\,(\lambda(B^w)^\top)=\pm\pi_w(\sigma)\ ,\ [\lambda]\in\mathbb{P}_{\mathbb{R}}^{n+k-d(\mathcal{A})-2}\setminus H_{\mathcal{A}}\right\}}$. There are no more than $S(n+2,k-2)$ of the first kind of intersection, thanks to Corollary 3.9. After applying the map $\pi_w$, we see that counting the second kind of intersections reduces to a lower-dimensional instance of Corollary 3.9. In particular, the second kind of intersections, for fixed $w$, contribute no more than $S(\dim(\mathcal{A}^w)+2,k_w-2)$ to our total, where $k_w$ is the number of columns of $\mathcal{A}^w$ minus $d(\mathcal{A}^w)$. Note that the sum of all the $k_w$ is no more than $k-1$ since $d(\mathcal{A})=n$. Note also that when $\mathcal{A}$ has just two non-simplicial facets, with one having exactly $n+1$ columns, the other has at most $n+k-4$ colums. In which case, these facets would contribute $S(n+1,0)+S(n+1,k-5)$ to our sum. In particular, this is the maximal possible contribution, over all distributions of points to the non-simplicial facets, thanks to Lemma 3.10.

More generally, the non-simplicial faces of $\mathcal{A}$ naturally form a poset under containment which, along with the distribution of the columns of $\mathcal{A}$ as points in the relative interior of the faces of $\text{Conv}\{a_1,\ldots,a_{n+k}\}$, determines the sum of $S(m,j)$ giving an upper bound for the intersection count we seek. Lemma 3.10 then tells us that our sum is maximized when it is of the form
$S(n+2,k-2)+(S(n+1,0)+S(n+1,k-5))+\cdots$
$\cdots+\left(S\left(n+2-\min\left\{n+1,\left\lfloor\frac{k-2}{3}\right\rfloor\right\},0\right)+S\left(n+2-\min\left\{n+1,\left\lfloor\frac{k-2}{3}\right\rfloor\right\},k-2-3\min\left\{n+1,\left\lfloor\frac{k-2}{3}\right\rfloor\right\}\right)\right).$
Since $S(m,0)=1$ for all $m$ we are done. ■

**Theorem 3.12.** [For17] *Let $[c_g]$ be any smooth point of $\Xi_{\mathcal{A}}$. Then $Z_{\mathbb{R}}(g)$ has a unique singular point $\zeta$, and the Hessian of $g$ at $\zeta$ has full rank.* ■

In what follows, let $N(g)$ denote the number of connected components of $Z_{\mathbb{R}}(g)$.

**Theorem 3.13.** [GPR17] *If $g$ is an $n$-variate exponential $(n+k)$-sum with spectrum $\mathcal{A}\in\mathbb{R}^{n\times(n+k)}$, and $(\text{Log}|c_g|)B$ lies in an outer chamber, then $N(g)\leq(n+k)(n+k-1)/2$.* ■

**Theorem 3.14.** *Suppose $n\geq 2$ and $g_-,g_*,g_+$ are $n$-variate exponential $(n+k)$-sums with non-defective spectrum $\mathcal{A}$, $\text{sign}(c_{g_-})=\text{sign}(c_{g_*})=\text{sign}(c_{g_+})=\sigma$, and $L'\subset\mathbb{R}^{k-1}$ is the unique line segment connecting $(\text{Log}|c_{g_-}|)B$ and $(\text{Log}|c_{g_+}|)B$. Suppose further that $L'\cap\widetilde{\Gamma}_\sigma(\mathcal{A},B)=\{(\text{Log}|c_{g_*}|)B\}$, and $(\text{Log}|c_{g_*}|)B$ is a smooth point of $\widetilde{\Gamma}_\sigma(\mathcal{A},B)$. Then $|N(g_+)-N(g_-)|\leq 1$ and $|N(g_\pm)-N(g_*)|\leq 1$.*

**Proof:** $X:=\{(c_g,y)\in\mathbb{R}^{n+k}\times\mathbb{R}^n \mid g(y)=0\ ,\ \text{sign}(c_g)=\sigma\ ,\ (\text{Log}|c_g|)B\in L'\}$ forms a singular real manifold but, thanks to Theorem 3.12, $X$ has a unique singularity at $(c_{g_*},\zeta)$ where $\zeta\in\mathbb{R}^n$ is the unique singular point of $Z_{\mathbb{R}}(g_*)$. Let $\phi:[-1,1]\longrightarrow\mathbb{R}^{n+k}$ be any smooth function with $\text{sign}(\phi(t))=\sigma$ for all $t\in[-1,1]$ and $(\text{Log}|\phi([-1,1])|)B=L'$. Let $\pi:\mathbb{R}^{n+k}\times\mathbb{R}^n\longrightarrow\mathbb{R}^{n+k}$ denote the natural orthogonal projection forgetting the second factor. We then see that $\phi^{-1}\circ\pi$ is a Morse function on $X$. By Stratified Morse Theory [Mil69, GM88], there is a closed ball $U\subset\mathbb{R}^{n+k}\times\mathbb{R}^n$ containing $(c_{g_*},\zeta)$ such that $U\cap X$ is homeomorphic to a real hypersurface of the form $Y=\{(x,t)\in\mathbb{R}^n\times[-1,1] \mid Q(x)=t,|x|\leq 1\}$, where $Q$ is a homogeneous quadratic form with signature identical to the Hessian of $g_*$ at $\zeta$, $Y\cap\{t=\pm 1\}$ is isotopic to $U\cap Z_{\mathbb{R}}(g_\pm)$, and $Y\cap\{t=0\}$ is isotopic to $U\cap Z_{\mathbb{R}}(g_*)$.

To conclude, observe that $Y\cap\{t=\pm 1\}$ empty implies that the signature of $Q$ is $\pm(1,\ldots,1)$, and thus $Y\cap\{t=0\}$ is a point and $Y\cap\{t=\mp 1\}$ is a sphere. So then $U\cap Z_{\mathbb{R}}(g_\pm)$ empty implies that $U\cap Z_{\mathbb{R}}(g_\mp)$ has a unique isolated connected component. In other words, the conclusion of our theorem is true.

If $Y\cap\{t=\pm 1\}$ are both non-empty, then the signature of $Q$ can *not* be $\pm(1,\ldots,1)$. So then $Y\cap\{t=-1\}$, $Y\cap\{t=0\}$, and $Y\cap\{t=1\}$, each have at least one connected component, and none has more than 2 connected components. This in turn implies that $U\cap Z_{\mathbb{R}}(g_-)$, $U\cap Z_{\mathbb{R}}(g_*)$, and $U\cap Z_{\mathbb{R}}(g_+)$ each have at least one connected component, and none has more than 2 connected components. Note also that any connected component of $U\cap Z_{\mathbb{R}}(g_\pm)$ (resp. $U\cap Z_{\mathbb{R}}(g_*)$) lies in a unique connected component of $Z_{\mathbb{R}}(g_\pm)$ (resp. $Z_{\mathbb{R}}(g_*)$). So we are done. ■

3.1. **The Proof of Theorem 1.2:** If $n=1$ then the theorem follows easily from the well-known generalization of Descartes' Rule of Signs to real exponents (see, e.g., [Wan04]), and with an improved (tight) upper bound of $k$. Note no genericity assumption is needed in this case: The bound holds for any nonzero $c_g\in(\mathbb{R}^*)^{1+k}$. So let us assume henceforth that $n\geq 2$.

**Combinatorially Simplicial Case:** If $\mathcal{A}$ is defective then $\Xi_{\mathcal{A}}\cap\mathbb{P}^{n+k-1}_{\mathbb{R}}$ has real codimension 2 in $\mathbb{P}^{n+k-1}_{\mathbb{R}}$ and thus $\mathbb{P}^{n+k-1}_{\mathbb{R}}\setminus\Xi_{\mathcal{A}}$ is path-connected. So then, by the framework of our proof of Theorem 3.14, the number of connected components of $g$ is constant for any fixed choice of sign vector. So it suffices to count connected components in outer chambers and, by Theorem 3.13, we are done. Note also that here, the genericity assumption arises from assuming that $[c_g]$ not lie in $\Xi_{\mathcal{A}}$. So let us now assume $\mathcal{A}$ is non-defective.

Consider a line segment $L_{gh}$, connecting $(\mathrm{Log}|c_g|)B$ to $(\mathrm{Log}|c_h|)B$, where $h$ has the same spectrum as $g$ and $\mathrm{sign}(c_h)=\mathrm{sign}(c_g)=:\sigma$, but known cardinality for $Z_{\mathbb{R}}(h)$. The key trick will then be that $L_{gh}$ intersects $\Gamma_\sigma(\mathcal{A},B)$ in few places, and the number of connected components of an $f$ with $\mathrm{Log}|c_f|\in L$ changes only slightly as $f$ moves from $h$ to $g$.

In particular, we may assume in addition that $h$ lies in an outer chamber $\mathcal{C}_\sigma$ (since outer chambers are open and unbounded). By Theorem 2.6 we may then assume that $L_{gh}$ lies in an affine line $L$ sufficiently generic for Corollary 3.9 to hold, *and* that $L_{gh}$ intersects $\Gamma_\sigma(\mathcal{A},B)$ only at smooth points of $\Gamma_\sigma(\mathcal{A},B)$. Furthermore, since the points of $L_{gh}\cap\Gamma_\sigma(\mathcal{A},B)$ can be linearly ordered, we may also assume that $(h,\mathcal{C}_\sigma)$ has been chosen so that $L_{gh}\cap\Gamma_\sigma(\mathcal{A},B)$ consists of no more than half of $L\cap\Gamma_\sigma(\mathcal{A},B)$.

If we can show that $Z_{\mathbb{R}}(h)$ has few connected components, and $Z_{\mathbb{R}}(f)$ gains few connected components as $f$ moves from $h$ to $g$ (with $\mathrm{Log}|c_f|$ restricted to $L$), then we'll be done.

Toward this end, observe that $Z_{\mathbb{R}}(h)$ has at most $(n+k)(n+k-1)/2$ connected components, thanks to Theorem 3.13. Since we have chosen $L_{gh}$ so that it intersects $\Gamma_\sigma(\mathcal{A},B)$ only at smooth points, Theorem 3.14 tells us that as $f$ moves from $h$ to $g$ (with $(\mathrm{Log}|c_f|)B$ restricted to $L$), each such intersection introduces at most 1 new connected component. (Theorems 3.1 also tell us that $N(f)$ is constant when $(\mathrm{Log}|c_f|)B$ lies between adjacent intersections in $L\cap\Gamma_\sigma(\mathcal{A},B)$.) So by Corollary 3.9, we are done with the case where $\mathcal{A}$ is combinatorially simplicial, with a slightly smaller upper bound of $\dfrac{(n+k)(n+k-1)}{2}+\lfloor S(n+2,k-2)/2\rfloor$. ■

**The Case Where $\mathcal{A}$ is not Combinatorially Simplicial:** Here we just slightly modify the argument we used when $\mathcal{A}$ was combinatorially simplicial: The key difference is that we work with $\widetilde{\Gamma}_\sigma(\mathcal{A},B)$ instead of $\Gamma_\sigma(\mathcal{A},B)$, and apply Corollary 3.11 instead of Corollary 3.9. (So here, the genericity condition arises from $[c_g]$ not lie in $\Xi_{\mathcal{A}}$ or any facial discriminant variety

$\Xi_{\mathcal{A}^w}$.) The number of intersections $L$ with $\widetilde{\Gamma}_\sigma(\mathcal{A},B)$ between $(\text{Log}|c_g|)B$ and $(\text{Log}|c_h|)B$ then clearly admits an upper bound of

$$T(n,k) := (S(n+2,k-2)+S(n+1,k-5)+ \cdots + S\left(n+2-\min\left\{n+1,\left\lfloor\frac{k-2}{3}\right\rfloor\right\}, k-2-3\min\left\{n+1,\left\lfloor\frac{k-2}{3}\right\rfloor\right\}\right) + \min\left\{n+1,\left\lfloor\frac{k-2}{3}\right\rfloor\right\})/2.$$

At this point, we are nearly done, but for some elementary observations on sums of powers of 2 and the size of $S(n+1,1)$. First, observe that the powers of 2 in the summands making up $T(n,k)$ are:

$$2^{(k-2)(k-3)/2}, 2^{(k-5)(k-6)/2}, \ldots, 2^{\left(k-2-3\min\left\{n+1,\left\lfloor\frac{k-2}{3}\right\rfloor\right\}\right)\left(k-3-3\min\left\{n+1,\left\lfloor\frac{k-2}{3}\right\rfloor\right\}\right)/2}.$$

So, in particular, the sum of all but the first power of 2 is strictly less than

$$2^{k^2-11k+30} + 2^{k^2-11k+29} + \cdots + 2^4 + 2^3 = 2^{k^2-11k+31} - 8 < 2^{(k-4)(k-5)/2} - 8.$$

Next, we observe that $\min\left\{n+1,\left\lfloor\frac{k-2}{3}\right\rfloor\right\} \leq n+1 < S(n+1,1)$. So then we easily obtain that $T(n,k) \leq (S(n+2,k-2)+S(n+1,k-4))/2$. So the final upper bound we obtain is $N(g) \leq \frac{(n+k)(n+k-1)}{2} + \lfloor (S(n+2,k-2)+S(n+1,k-4))/2 \rfloor$, which is slightly better than our stated bound. ■

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