$\mathcal{A}$-DISCRIMINANTS FOR COMPLEX EXPONENTS, AND COUNTING REAL ISOTOPY TYPES

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Abstract. We extend the definition of $\mathcal{A}$-discriminant varieties, and Kapranov's parametrization of $\mathcal{A}$-discriminant varieties, to complex exponents. As an application, we study the special case where $\mathcal{A}$ is a fixed real $n \times (n+3)$ matrix whose columns form the spectrum of an n-variate exponential sum g with fixed signed vector for its coefficients: We prove that the number of possible isotopy types for the real zero set Z of g, when Z is smooth, is $O(n^2)$. The best previous upper bound was $2^{O(n^4)}$. Along the way, we also prove (a) the singular loci of our generalized $\mathcal{A}$-discriminant varieties are images of low-degree algebraic sets under certain analytic maps, and (b) an extension of a classical result of Steiner (on the number of cells in a planar line arrangement) to arrangements of curves with cusps and poles.

1. Introduction

Classifying families of real algebraic plane curves up to deformation is part of Hilbert's 16^{th} Problem and, for curves of high degree, remains a challenging open problem (see, e.g., [Kal03, OK03]). While the number of possible isotopy types is now known to be exponential in the degree, degree is but one measure of the complexity of a polynomial. Here, in the spirit of Khovanski's Fewnomial Theory [Kho91], we will see an example of a different measure of complexity of zero sets, as well as a different class of families for which the number of isotopy types admits a polynomial upper bound. The techniques we use to achieve these bounds may be of independent interest.

In particular, we take a first step toward extending the theory of $\mathcal{A}$-discriminants [GKZ94], and Kapranov's parametrization of $\mathcal{A}$-discriminant varieties [Kap91], to a broader family of analytic functions. As an application, we prove a quadratic upper bound on the number of isotopy types of real zero sets of certain n-variate exponential sums, in a setting where the best previous bounds were exponential in n^4. In a companion paper [FNR17], we apply the techniques developed here to give new polynomial upper bounds on the number of connected components of such real zero sets.

More precisely, let $\mathcal{A} = [a_{i,j}] \in \mathbb{C}^{n\times t}$ have distinct columns, with j^{th} column a_j, $y := (y_1, \ldots, y_n) \in \mathbb{C}^n$, $a_j \cdot y := \sum_{i=1}^n a_{i,j} y_i$, and consider exponential sums of the form

$$g(y) := \sum_{j=1}^{t} c_j e^{a_j \cdot y},$$

where $c_j \in \mathbb{C} \setminus \{0\}$ for all j. We call g an *n-variate exponential t-sum*, $\mathcal{A}$ the *spectrum* of g, set $c_g := (c_1, \ldots, c_t)$, and call each a_j a *frequency*. Paralleling the usual notation of affine algebraic sets, we let $Z_{\mathbb{R}}(g)$ (resp. $Z_{\mathbb{C}}(g)$) denote the set of roots of g in $\mathbb{R}^n$ (resp. $\mathbb{C}^n$). In particular, the notations $Z_{\mathbb{R}}(g)$ and $Z_{\mathbb{C}}(g)$ imply that we consider g as a function on $\mathbb{C}^n$, and the spectrum and coefficient vector of g to be fixed. We call g a *real* exponential sum when both $\mathcal{A} \in \mathbb{R}^{n\times t}$ and all the coefficients c_j of g are real. Also, given any two subsets $X, Y \subseteq \mathbb{R}^n$, an *isotopy from X to Y (ambient in $\mathbb{R}^n$)* is a continuous map $I : [0,1] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfying (1) $I(t, \cdot) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a homeomorphism for each $t \in [0,1]$, (2) $I(0,x) = x$ for all $x \in \mathbb{R}^n$,

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and (3) $I(1,X)=Y$.[1] Although our generalized $\mathcal{A}$-discriminants allow arbitrary $\mathcal{A}\in\mathbb{C}^{n\times t}$, our isotopy counts will focus on smooth $Z_{\mathbb{R}}(g)$, in the special case where $c_j\in\mathbb{R}\setminus\{0\}$ for all j and we fix both $\mathcal{A}\in\mathbb{R}^{n\times(n+3)}$ and $\mathrm{sign}(c_g):=(\mathrm{sign}(c_1),\ldots,\mathrm{sign}(c_t))$ (the *sign vector* of g).

A consequence of our generalized $\mathcal{A}$-discriminants (defined in the next section, and parametrized explicitly in Definition 1.6 and Theorem 1.7 below) is the following new count of isotopy types:

Theorem 1.1. *Following the notation above, assume $t=n+3$, $\{a_1,\ldots,a_{n+3}\}$ does not lie in any affine hyperplane, and that we fix $\mathcal{A}:=[a_1,\ldots,a_{n+3}]$ and $\sigma\in\{\pm1\}^{n+3}$. Then the number of possible isotopy types for $Z_{\mathbb{R}}(g)$, over all $c_g\in(\mathbb{R}\setminus\{0\})^{n+3}$ with $\mathrm{sign}(c_g)=\sigma$ and $Z_{\mathbb{R}}(g)$ smooth, is no greater than $2n^2+11n+16$.*

We prove Theorem 1.1 in Section 6. One technicality to observe is that our proof does not rule out a new collection of isotopy types for $Z_{\mathbb{R}}(g)$ if we consider a new value for $\mathrm{sign}(c_g)$, or if we consider a new $(n+3)$-tuple of frequencies. Hence our assumption that $\mathcal{A}$ and $\mathrm{sign}(c_g)$ be fixed. There is evidence (from the special case $t=n+2$ [BPRRR18]) that the total number of isotopy types should still be polynomial in n even if we allow both $\mathcal{A}$ and $\mathrm{sign}(c_g)$ to vary, but we will leave these extensions for future work.

Using the abbreviation $x^u:=x_1^{u_1}\cdots x_n^{u_n}$ when $(u_1,\ldots,u_n)\in\mathbb{Z}^n$, we call $f(x):=\sum_{j=1}^t c_jx^{a_j}$ an *n-variate t-nomial* when $\mathcal{A}\in\mathbb{Z}^{n\times t}$. The change of variables $x=e^y:=(e^{y_1},\ldots,e^{y_n})$ shows that, when $\mathcal{A}\in\mathbb{Z}^{n\times t}$, studying zero sets of t-nomials in the positive orthant $\mathbb{R}^n_+$ is the same as studying zero sets of exponential t-sums in $\mathbb{R}^n$, up to the diffeomorphism between $\mathbb{R}^n_+$ and $\mathbb{R}^n$ defined by $x=e^y$. So in this sense, dealing with exponential sums is a generalization of the polynomial case.

A result of Basu and Vorobjov [BV07] implies a $2^{O(t^4)}(nt)^{O(t(t+n))}$ upper bound for the number of isotopy types of a smooth $Z_{\mathbb{R}}(g)$ when g is an n-variate t-sum. Also, although [DRRS07, Thm. 1.3] presents an $O(n^6)$ upper bound for the special case $\mathcal{A}\in\mathbb{Z}^{n\times(n+3)}$ (with some additional restrictions), Theorem 1.1 here is sharper and gives the first polynomial upper bound on the number of isotopy types in the setting of n-variate exponential $(n+3)$-sums.

Recall that for any subset $S\subseteq\mathbb{R}^n$, its *convex hull*, denoted $\mathrm{Conv}S$, is the smallest convex set containing S. It is important to observe that counting isotopy types becomes more complicated when $\{a_1,\ldots,a_t\}$ lies in an affine hyperplane: Lemma 1.2 below reveals that the number of isotopy types for smooth $Z_{\mathbb{R}}(g)$ with underlying spectrum $\mathcal{A}=[a_1,\ldots,a_t]$ depends on the *dimension of* $\mathrm{Conv}\{a_1,\ldots,a_t\}$, as well as its cardinality. In what follows, we will use the standard computer science notations for bounds holding asymptotically up to a constant multiple: $O(\cdot)$, $\Omega(\cdot)$, and $\Theta(\cdot)$ [CLRS09].

Lemma 1.2. *There exists a family of spectra $\{\mathcal{A}_t\}_{j\in\mathbb{N}}$, with $\mathcal{A}_t\in\mathbb{Z}^{1\times t}$ (resp. $\mathcal{A}_t\in\mathbb{Z}^{2\times t}$) for all t, with the following property: The number of possible isotopy types of $Z_{\mathbb{R}}(g)$, over all g with spectrum $\mathcal{A}_t$, $c_g\in(\mathbb{R}\setminus\{0\})^t$, and $Z_{\mathbb{R}}(g)$ smooth, is t (resp. $2^{\Omega(t)}$).*

We prove Lemma 1.2 in Section 6.1. Letting $d(\mathcal{A}):=\dim\mathrm{Conv}\{a_1,\ldots,a_t\}$, note that $1\leq d(\mathcal{A})\leq\min\{n,t-1\}$ when $t\geq2$, and $Z_{\mathbb{R}}(g)$ is empty when $t=1$. Note also that

[1] Letting J be an isotopy from Y to Z, and setting $i_t(x):=I(t,x)$ and $j_t(x):=J(t,x)$, it is easy to see that $j_t(i_t(x))$ defines an isotopy from X to Z, and that i_t^{-1} (i.e., the inverse of the bijection $i_t:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ for any fixed t) defines an isotopy from Y to X. So isotopy is in fact an equivalence relation and it makes sense to speak of *isotopy type*.

the cases where $t-d(\mathcal{A})$ is fixed are *not* addressed by Lemma 1.2, since $d(\mathcal{A}_t)\leq 2$ for all t in Lemma 1.2. In particular, for any fixed $\mathcal{A}$ with $t-d(\mathcal{A})\leq 2$, and any fixed $\sigma\in\{\pm 1\}^t$, the number of possible isotopy types for $Z_{\mathbb{R}}(g)$, over all $c_g\in(\mathbb{R}\setminus\{0\})^t$ with $\operatorname{sign}(c_g)=\sigma$ and $Z_{\mathbb{R}}(g)$ smooth, is known to be 2 (see, e.g., [PRT09, Bih11]). Our Theorem 1.1 thus addresses the case $t-d(\mathcal{A})=3$. Let us now introduce our main theoretical tools.

1.1. Generalizing, and Parametrizing, $\mathcal{A}$-Discriminants for Complex Exponents.

Our isotopy count provides a motivation for generalized $\mathcal{A}$-discriminants, since discriminants parametrize degenerate behavior, and different isotopy types can be obtained by deforming a zero set through a degenerate state. This connection is classical and well-known. See, for instance, Hardt's Triviality Theorem [Har80] (in the semi-algebraic setting) or [GKZ94, Ch. 11, Sec. 5] (in the real algebraic setting): The connected components of the complement of the real part of a discriminant variety describe regions in coefficient space (called *discriminant chambers*) where the topology of the real zero set (in a suitable compactification of $\mathbb{R}^n$) of a polynomial is constant. Our central object of study will be the following kind of discriminant variety associated to families of exponential sums:

Definition 1.3. *Let $\mathcal{A}\in\mathbb{C}^{n\times t}$ have $j^{\underline{th}}$ column a_j and assume the columns of $\mathcal{A}$ are distinct. We then define the* generalized $\mathcal{A}$-discriminant variety, *$\Xi_{\mathcal{A}}$, to be the Euclidean closure of the set* $\left\{\,[c_1:\cdots:c_t]\in\mathbb{P}^{t-1}_{\mathbb{C}}\setminus\{c_1\cdots c_t\neq 0\}\ \middle|\ \sum_{j=1}^{t} c_j e^{a_j\cdot y} \text{ has a degenerate root } y\in\mathbb{C}^n\right\}$. ◇

When $\mathcal{A}\in\mathbb{Z}^{n\times t}$ our $\Xi_{\mathcal{A}}$ agrees with the classical $\mathcal{A}$-discriminant variety $\nabla_{\mathcal{A}}$: This follows immediately from the definition of $\nabla_{\mathcal{A}}$ (see, e.g., [GKZ94, Ch. 9, Sec. 1, pg. 271]). More to the point, for $t-d(\mathcal{A})\geq 3$, the algebraic hypersurface $\nabla_{\mathcal{A}}$ has a defining polynomial that is often too unwieldy for computational purposes. So we will also need a more efficient, alternative description for $\Xi_{\mathcal{A}}$.

Example 1.4. *Let* $\mathcal{A}=\begin{bmatrix}0&1&0&4&1\\0&0&1&1&4\end{bmatrix}$. *Then $t=5$, $d(\mathcal{A})=2$, and a routine* `maple` *calculation shows us that $\nabla_{\mathcal{A}}$ (and thus $\Xi_{\mathcal{A}}$) is exactly the zero set in $\mathbb{P}^4_{\mathbb{C}}$ of a homogeneous polynomial $\Delta_{\mathcal{A}}\in\mathbb{Z}[c_1,\ldots,c_5]$ satisfying $\Delta_{\mathcal{A}}(1,1,1,a,b)=$*

$$
\begin{aligned}
&419876545047715235939922227a^8b^8+856892261757779082796 0320a^8b^7+394594247668399678957787136a^7b^8+\\
&491069384583950065193975808a^8b^6-971141005960243113814917120a^7b^7+164454681104805909036662784 0a^6b^8+\\
&557969223231079901560832a^9b^4+82843494158262383800850841 6a^8b^5-889611814368712453728606617 6a^7b^6+\\
&469208414291313561986872115 2a^6b^7+284549969837299986680984371 2a^5b^8+557969223231079901560832a^4b^9+\\
&333929965005366315555225 60a^9b^3+384254443547034707078152192a^8b^4-114834435026445610699099996 16a^7b^5+\\
&143231076647749243489799372 80a^6b^6+135910000630336852710549094 40a^5b^7+222567667963172933995593728 0a^4b^8-\\
&255112835673284571949957 12a^3b^9+20941053496075364622925824a^9b^2+269737322421295126029533184a^8b^3-\\
&251455812374364457158049792 0a^7b^4+213192821214309821869635665 92a^6b^5+181381633163744066595276718 08a^5b^6+\\
&459434896114086755201292697 6a^4b^7+382889518651229479821639 68a^3b^8-5152464593144578003540377 6a^2b^9+\\
&36308726360282510445 7728a^{10}-97920096402886895358443 52a^9b+178810349707236426746167296a^8b^2+\\
&136826425411721658954783129 6a^7b^3+103972479521860847665906974 72a^6b^4+693072660882072549290567270 4a^5b^5+\\
&253511942255388095089213440 0a^4b^6+134703665565747736152637440a^3b^7-92973237722754317832683520a^2b^8-\\
&2893351631835012551147520ab^9+363087263602825104457728b^{10}+726174527205650208915456a^9-\\
&12696707749111290371506176a^8b-744896214235170878364057 60a^7b^2+363189381895713399018356736a^6b^3-\\
&482236618449489680142434304a^5b^4+191290255533750888626651136a^4b^5+505712479336809840802529 28a^3b^6-\\
&312822370547809004059361 28a^2b^7-5798049740657613386809344ab^8+726174527205650208915456b^9+\\
&363087263602825104457728a^8-2904698108822600835661824a^7b+10166443380879102924816384a^6b^2-\\
&20332886761758205849632768a^5b^3+25416108452197757312040960a^4b^4-20332886761758205849632768a^3b^5+\\
&10166443380879102924816384a^2b^6-2904698108822600835661824ab^7+363087263602825104457728b^8.\ \diamond
\end{aligned}
$$

The *generalized* $\mathcal{A}$-discriminant need not be the zero set of *any* polynomial function, already for $\mathcal{A}\in\mathbb{R}^{1\times 3}\setminus\mathbb{Q}^{1\times 3}$: For instance, taking $\mathcal{A}=\left[0,1,\sqrt{2}\right]$, it is not hard to check that the intersection of the $c_1=c_3$ line with $\Xi_{\mathcal{A}}$ in $\mathbb{P}^2_{\mathbb{C}}$ is exactly the infinite set

$$\left\{\left[1 : \frac{-\sqrt{2}}{\sqrt{2}-1}(\sqrt{2}-1)^{1/\sqrt{2}}e^{\sqrt{-2}\pi k} : 1\right] \;\middle|\; k\in\mathbb{Z}\right\}.$$

So this particular $\Xi_{\mathcal{A}}$ can't even be semi-algebraic.[2]

Nevertheless, the generalized discriminant $\Xi_{\mathcal{A}}$ admits a concise and explicit parametrization that will be our main theoretical tool. Some more notation we'll need is the following.

Definition 1.5. *For any $\mathcal{A}\in\mathbb{C}^{n\times t}$ let $\widehat{\mathcal{A}}\in\mathbb{C}^{(n+1)\times t}$ denote the matrix with first row $[1,\ldots,1]$ and bottom n rows forming $\mathcal{A}$, and let $B\in K^{t\times(t-d(\mathcal{A})-1)}$ be any matrix whose columns form a basis for the right nullspace of $\widehat{\mathcal{A}}$, where K is the minimal field containing the entries of $\mathcal{A}$. Let β_i denote the $i^{\underline{th}}$ row of B. Finally, let us call $\mathcal{A}$* non-defective *if we also have that (0) the columns of $\mathcal{A}$ are distinct and (1)* $\mathrm{codim}_{\mathbb{C}}\Xi_{\mathcal{A}}=1$. $\diamond$

It is easily checked $d(\mathcal{A})=\left(\mathrm{rank}\widehat{\mathcal{A}}\right)-1$. In particular, the existence of a $B\in K^{t\times(t-n-1)}$ with columns forming a basis for the right nullspace of $\widehat{\mathcal{A}}$ is equivalent to $\widehat{\mathcal{A}}$ having rank $n+1$. Note also that $\mathcal{A}$ can be defective when $d(\mathcal{A})=n$: For instance, $\mathcal{A}=\begin{bmatrix}0&1&0&0&1&0\\0&0&1&0&0&1\\0&0&0&1&1&1\end{bmatrix}$ has $d(\mathcal{A})=n=3$ and $t=6$ but is defective (see, e.g., [BHPR11, Ex. 2.8 & Cor. 3.7] or [DS02, DR06]). It is useful to note, however, that a generic $\mathcal{A}$ is non-defective: See, for instance, [For18, Thm. 6.4].

We will be able to write down our parametrization of $\Xi_{\mathcal{A}}$ once we define a particular hyperplane arrangement.

Definition 1.6. *Let $(\cdot)^\top$ denote matrix transpose and, for any $u:=(u_1,\ldots,u_N)$ and $v:=(v_1,\ldots,v_N)$ in $\mathbb{C}^N$, let $u\odot v:=(u_1v_1,\ldots,u_Nv_N)$. Then, following the notation of Definition 1.5, assume $\mathcal{A}$ is non-defective, set $\lambda:=(\lambda_1,\ldots,\lambda_{t-d(\mathcal{A})-1})$ and $[\lambda]:=[\lambda_1:\cdots:\lambda_{t-d(\mathcal{A})-1}]$. We then define the (projective) hyperplane arrangement*

$$H_{\mathcal{A}}:=\{[\lambda] \mid \lambda\cdot\beta_i=0 \text{ for some } i\in\{1,\ldots,t\}\}\subset\mathbb{P}^{t-d(\mathcal{A})-2}_{\mathbb{C}},$$

and define $\psi_{\mathcal{A}}:\left(\mathbb{P}^{t-d(\mathcal{A})-2}_{\mathbb{C}}\setminus H_{\mathcal{A}}\right)\times\mathbb{C}^n\longrightarrow\mathbb{P}^{t-1}_{\mathbb{C}}$ *by* $\psi_{\mathcal{A}}([\lambda],y):=\left[(\lambda B^\top)\odot e^{-y\mathcal{A}}\right]$. $\diamond$

While $\psi_{\mathcal{A}}$ certainly depends on B, its image is easily seen to be independent of B: Simply note that, up to transposes, $\left\{\lambda B^\top\right\}_{\lambda\in\mathbb{P}^{t-d(\mathcal{A})-2}_{\mathbb{C}}}$ is the right-null space of $\widehat{\mathcal{A}}$. More importantly, as we'll soon see, $\psi_{\mathcal{A}}$ is in fact a parametrization of $\Xi_{\mathcal{A}}$. In what follows, for any fixed choice of $\sigma\in\{\pm1\}^t$, we call $\mathbb{P}^{t-1}_{\sigma}:=\{[\lambda]\in\mathbb{P}^{t-1}_{\mathbb{R}} \mid \mathrm{sign}(\lambda)=\pm\sigma\}$ an *orthant* of $\mathbb{P}^{t-1}_{\mathbb{R}}$. We also let $\mathrm{Log}|(z_1,\ldots,z_N)|:=(\log|z_1|,\ldots,\log|z_n|)$ denote the coordinate-wise log-absolute value map.

Theorem 1.7. *If $\mathcal{A}\in\mathbb{C}^{n\times t}$ is non-defective then:*

1. *$\Xi_{\mathcal{A}}$ is exactly the Euclidean closure of the analytic hypersurface $\psi_{\mathcal{A}}\left(\left(\mathbb{P}^{t-d(\mathcal{A})-2}_{\mathbb{C}}\setminus H_{\mathcal{A}}\right)\times\mathbb{C}^n\right)$. In particular, $\Xi_{\mathcal{A}}$ is a connected subset of $\mathbb{P}^{t-1}_{\mathbb{C}}$.*
2. *If we also have $\mathcal{A}\in\mathbb{R}^{n\times t}$, and $\sigma\in\{\pm1\}^t$, then $\mathrm{Log}|\Xi_{\mathcal{A}}\cap\mathbb{P}^{t-1}_{\sigma}|$ is an $\mathbb{R}^n$-bundle over a base of the form $\Gamma_\sigma(\mathcal{A})\cup Y_\sigma\subset\mathbb{R}^{t-d(\mathcal{A})-1}$ where*
   a. *$\Gamma_\sigma(\mathcal{A})$ is the closure of the union of $O(t)^{4t-2d(\mathcal{A})-2}$ real analytic hypersurfaces.*
   b. *Y_σ is a countable, locally finite union of real analytic varieties of codimension ≥ 2.*

We prove Theorem 1.7 in Section 4. The special case $\mathcal{A}\in\mathbb{Z}^{n\times t}$ of the first assertion recovers the famous *Horn-Kapranov Uniformization*, derived by Kapranov in [Kap91]. We call the

[2]Note that while Serre's famous GAGA Theorem [Ser56] implies that $\Xi_{\mathcal{A}}$ is (analytically) isomorphic to a complex projective algebraic variety, $\Xi_{\mathcal{A}}$ is not necessarily *equal* to a complex projective algebraic variety: Our example shows that the underlying isomorphism can be a non-trivial analytic map.

$\Gamma_\sigma(\mathcal{A})$ *signed reduced contours*. We will see in Section 2 how signed reduced contours can be used to detect changes in isotopy type and, in the setting of Theorem 1.1, signed reduced contours are plane curves.

Theorem 1.7 is the first key idea toward proving Theorem 1.1. The remaining work is proving that counting isotopy types can be reduced to counting the number of connected components of the complement of a well-behaved union of convex arcs. This is detailed in Sections 2, 3, and 6 below. We also make use of a quantitative estimate on arrangements of plane curves, possibly of independent interest, detailed below.

1.2. **Counting Regions Determined by Complements of Unions of Curves with Few Cusps.** In 1826, Steiner studied line arrangements in $\mathbb{R}^2$ and proved that m lines determine no more than $\frac{m(m-1)}{2}+m+1$ connected components for the complement of their union in $\mathbb{R}^2$ [Ste26]. Following the development of the second author's Ph.D. thesis (see Theorems 3.6 and 3.7 of [Rus13]), we generalize Steiner's result below. In what follows, a *piece-wise smooth arc with cusps and poles* is the image C of a function $\varphi : [0,1] \longrightarrow \mathbb{R}^2$ that is differentiable on $[0,1]\setminus E$ for some finite E and is continuous at every s where $\varphi(s)$ is well-defined. So $\varphi(s)$ can be unbounded for only finitely many $s\in[0,1]$, and each such s is called a *pole*. An $s\in[0,1]$ where φ is non-differentiable (but $\varphi(s)$ is well-defined) is called a *cusp*. Finally, we call C *closed* if $\varphi(0)=\varphi(1)$.

Theorem 1.8. *Suppose $C\subset\mathbb{R}^2$ is a closed, piece-wise smooth arc with exactly ℓ cusps and m poles. Suppose also that no two distinct smooth sub-arcs of C intersect more than once, and that sub-arcs sharing a cusp or a pole do not intersect. Then C has at most $\binom{\ell+m}{2}-(\ell+m)$ self-intersections, and the complement $\mathbb{R}^2\setminus C$ has at most $\frac{(\ell+m)(\ell+m-1)}{2}-(\ell+m)+1$ connected components.*

We prove Theorem 1.8 in Section 5 below.

A key step in our proof of Theorem 1.1 will be to prove that reduced signed contours (for $t-d(\mathcal{A})=3$) are always a union of convex arcs of the form C above. This will imply that the complement of the real part of $\Xi_\mathcal{A}$ has few connected components. We now describe why this is central to counting isotopy types.

2. Defining Generalized $\mathcal{A}$-Discriminant Contours and Chambers

Let us first observe some basic cases where $\Xi_\mathcal{A}$, or at least its real part, admits a succinct description.

Lemma 2.1. *Assume $\mathcal{A}\in\mathbb{C}^{n\times t}$ has distinct columns. Then:*
1. *If $t-d(\mathcal{A})=1$ then $\Xi_\mathcal{A}=\emptyset$.*
2. *If $\mathcal{A}\in\mathbb{R}^{n\times t}$, $t-d(\mathcal{A})=2$, and $b=(\beta_1,\ldots,\beta_t)\in(\mathbb{R}\setminus\{0\})^t$ is a generator for the right nullspace of $\widehat{\mathcal{A}}$, then $[c_1:\cdots:c_t]\in\Xi_\mathcal{A}\cap\mathbb{P}^{t-1}_\mathbb{R}$ is equivalent to the following condition:*
$$\prod_{j=1}^t\left|\frac{c_j}{\beta_j}\right|^{\beta_j}=1 \text{ and } \operatorname{sign}(c_1,\ldots,c_t)=\pm\operatorname{sign}(b).$$
In particular, $\Xi_\mathcal{A}\cap\mathbb{P}^{t-1}_\sigma$ is non-empty exactly when $\operatorname{sign}(b)=\pm\sigma$.

Proof: The first assertion follows from two observations: (a) We may assume $a_1=\mathbf{O}$ since $\Xi_\mathcal{A}$ is invariant under translation of the point set $\{a_1,\ldots,a_{n+1}\}$ and (b) we can apply the invertible change of variables $y=[a_2,\ldots,a_{n+1}]^{-1}z$ to reduce to the case where $g(z)=c_1+c_2e^{z_1}+\cdots+c_{n+1}e^{z_n}$, which clearly yields $\Xi_\mathcal{A}=\emptyset$.

For the second assertion, the special case of integral $\mathcal{A}$ was observed earlier in [GKZ94, Prop. 1.8, Pg. 274]. More generally, we simply observe from first principles that g has a degenerate root at y if and only if $[c_1e^{a_1\cdot y},\ldots,c_te^{a_t\cdot y}]$ is a non-zero multiple of b. The product and sign conditions on c_g then immediately follow from the last equality.

Conversely, if the product and sign conditions on c_g hold, then we can solve for a degenerate root $y\in\mathbb{C}^n$ of g as follows: Without loss of generality, we may assume (by translation invariance again) that $a_1=\mathbf{O}$. Consider then $y=\left(\mathrm{Log}\left|[a_2,\ldots,a_{n+1}]^{-1}\left[\frac{\beta_2/\beta_1}{c_2/c_1},\ldots,\frac{\beta_{n+1}/\beta_1}{c_{n+1}/c_1}\right]^\top\right|\right)^\top$: The product and sign conditions then imply that y is in fact a degenerate root of g. ■

Let us now observe a simple property of non-defective $\mathcal{A}$. First, let us call $\mathcal{A}\in\mathbb{R}^{n\times t}$ *pyramidal* if and only if $\mathcal{A}$ has a column a_j such that $\{a_1,\ldots,a_{j-1},a_{j+1},\ldots,a_t\}$ lies in a $(d(\mathcal{A})-1)$-dimensional affine subspace.

Proposition 2.2. *Following the preceding notation, $\mathcal{A}$ is pyramidal if and only if B has a zero row. In particular, $\mathcal{A}$ non-defective implies that $\mathcal{A}$ is not pyramidal.*

Proof: Observe that a column $b_i=[b_{1,i},\ldots,b_{t,i}]^\top$ of B has $b_{j,i}\neq 0$ if and only if a_j lies in the same affine hyperplane as at least one of $a_1,\ldots,a_{j-1},a_{j+1},\ldots,a_t$. By construction, the affine relations defined by $b_1,\ldots,b_{t-d(\mathcal{A})-1}$ are a basis for *all* affine relations of the a_ℓ, and the affine span of the a_ℓ has dimension $d(\mathcal{A})$.

Now, if $a_1,\ldots,a_{j-1},a_{j+1},\ldots,a_t$ do *not* lie in any $(d(\mathcal{A})-1)$-dimensional affine subspace, then $a_1,\ldots,a_{j-1},a_{j+1},\ldots,a_t$ would have affine span of dimension $d(\mathcal{A})$ and thus a_j would have to lie in the affine span of $a_1,\ldots,a_{j-1},a_{j+1},\ldots,a_t$. So $b_{j,1}=\cdots=b_{j,t-d(\mathcal{A})-1}=0$ would be impossible. Conversely, if $a_1,\ldots,a_{j-1},a_{j+1},\ldots,a_t$ lie in some d-dimensional affine subspace with $d\leq d(\mathcal{A})-1$, then there are $(t-1)-d-1=t-d-2\geq t-d(\mathcal{A})-1$ linearly independent affine relations for $a_1,\ldots,a_{j-1},a_{j+1},\ldots,a_t$. So we must have $b_{j,1}=\cdots=b_{j,t-d(\mathcal{A})-1}=0$, and the first assertion is proved.

To prove the second assertion, assume $\mathcal{A}$ is pyramidal. Observing that $\Xi_\mathcal{A}$ is invariant under invertible linear maps applied to $\mathcal{A}$, and permuting coordinates if necessary, we may assume $j=t$, $a_t=e_n$ (the $n^{\underline{\text{th}}}$ standard basis vector), and $a_1,\ldots,a_{t-1}\in\mathbb{R}^{n-1}\times\{0\}$. In particular, we then see that $\Xi_\mathcal{A}$ is simply $\Xi_{\{a_1,\ldots,a_{t-1}\}}\times\{0\}$, and thus $\dim\Xi_\mathcal{A}<t-1$. ■

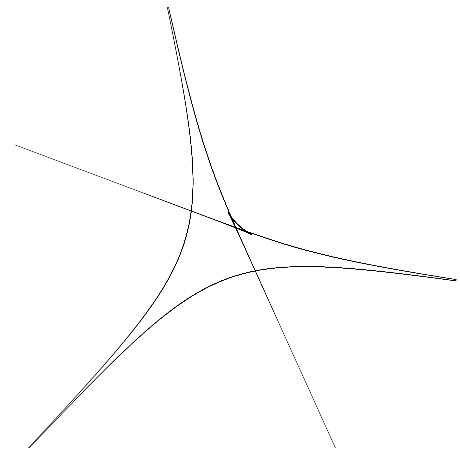
Since singularities in $Z_\mathbb{C}(g)$ are preserved under translation of $\mathcal{A}$ and scaling the coefficients of g, the discriminant variety $\Xi_\mathcal{A}$ possesses certain homogeneities that we can quotient out to simplify our study of discriminant chambers. Taking $\mathrm{Log}|\cdot|$ helps clarify these homogeneities and, as we'll see at the end of this section, also helps clarify the geometry of the underlying quotient. Toward this end, observe that $[1,\ldots,1]B=\mathbf{O}$. Clearly then, for any $\ell\in(\mathbb{R}\setminus\{0\})^t$, we have that $(\mathrm{Log}|\eta\ell|)B$ is independent of η for any nonzero $\eta\in\mathbb{C}\setminus\{0\}$, so we can then define $(\mathrm{Log}|[\ell]|)B:=(\mathrm{Log}|\ell|)B$. In what follows, for any $S\subseteq\mathbb{C}^N$ we let $\overline{S}$ denote the Euclidean closure of S.

Definition 2.3. *Following the notation of Definition 1.6, define*

$$\xi_{\mathcal{A},B}:\left(\mathbb{P}^{t-d(\mathcal{A})-2}_\mathbb{C}\setminus H_\mathcal{A}\right)\longrightarrow\mathbb{R}^{t-d(\mathcal{A})-1}\quad\text{by}\quad\xi_{\mathcal{A},B}([\lambda]):=\left(\mathrm{Log}\left|\lambda B^\top\right|\right)B.$$

(So $\xi_{\mathcal{A},B}$ is defined by multiplying a row vector by a matrix.) We then define the reduced discriminant contour*, $\Gamma(\mathcal{A},B)$, to be $\emptyset$ or $\overline{\xi_{\mathcal{A},B}\left(\mathbb{P}^{t-d(\mathcal{A})-2}_\mathbb{R}\setminus H_\mathcal{A}\right)}$, according as $\mathcal{A}$ is defective or not.* $\diamond$

Example 2.4. *When* $\mathcal{A} := \begin{bmatrix} 0 & 1 & 0 & 4 & 1 \\ 0 & 0 & 1 & 1 & 4 \end{bmatrix}$ *we are working with the family of exponential sums* $g(y) := f(e^{y_1}, e^{y_2})$ *where* $f(x) = c_1 + c_2x_1 + c_3x_2 + c_4x_1^4x_2 + c_5x_1x_2^4$. *A suitable* B *(among many others) with columns defining a basis for the right nullspace of* $\widehat{\mathcal{A}}$ *is then* $B \approx \begin{bmatrix} 0.5079 & -0.8069 & 0.1721 & 0.2267 & -0.0997 \\ 0.5420 & 0.1199 & -0.7974 & -0.0851 & 0.2206 \end{bmatrix}^{\top}$, *and the corresponding reduced contour* $\Gamma(\mathcal{A}, B)$, *intersected with* $[-4, 4]$, *is drawn to the right.* $\diamond$



Note that in our preceding example, $\Xi_{\mathcal{A}}$ is a hypersurface in $\mathbb{P}^4_{\mathbb{C}}$, and $Z_{\mathbb{C}}(g)$ has a singular point $y \in \mathbb{C}^2$ if and only if $Z_{\mathbb{C}}(h)$ has singular point $y + (\delta_1, \delta_2)$, where $h(y) := \alpha g(y_1 - \delta_1, y_2 - \delta_2)$ for some $\alpha \in \mathbb{C}^*$ and $\delta_i \in \mathbb{R}$. It is then easily checked that $\text{Log}|\Xi_{\mathcal{A}} \cap \mathbb{P}^4_{\mathbb{R}}|$ is a 2-plane bundle over the curve in $\mathbb{R}^2$ drawn above. We also point out that our choice of B above is merely an artifact of `Matlab`'s default of using an orthonormal basis when computing nullspaces: Up to an invertible linear map on $\mathbb{R}^2$, any other B would have given the same reduced contour above.

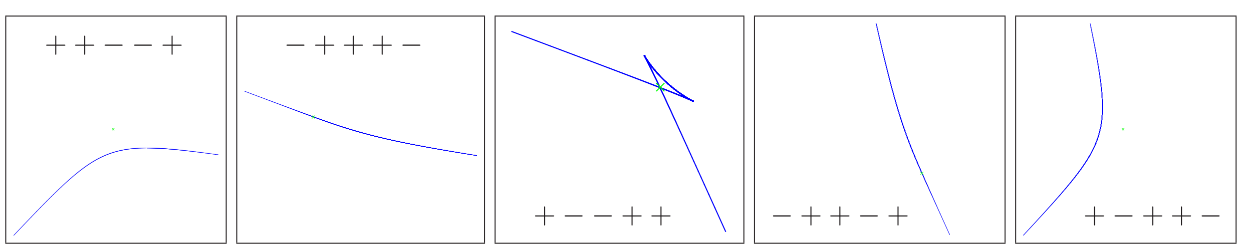
An obvious issue behind taking $\text{Log}|\cdot|$ of the real part of $\Xi_{\mathcal{A}}$ is that we lose information about coefficient signs. So we refine the notion of discriminant contour as follows:

Definition 2.5. *Suppose* $\mathcal{A} \in \mathbb{R}^{n \times t}$ *is non-defective and* $\sigma = (\sigma_1, \ldots, \sigma_t) \in \{\pm 1\}^t$. *We then define the* signed *reduced discriminant contour,* $\Gamma_\sigma(\mathcal{A}, B)$, *to be* $\emptyset$ *or the Euclidean closure of*

$$\left\{ \xi_{\mathcal{A},B}([\lambda]) \mid \text{sign}\left(\lambda B^{\top}\right) = \pm\sigma \ , \ [\lambda] \in \mathbb{P}^{t-d(\mathcal{A})-2}_{\mathbb{R}} \setminus H_{\mathcal{A}} \right\} \subset \mathbb{R}^{t-d(\mathcal{A})-1},$$

according as $\mathcal{A}$ *is defective or not. We call any connected component* $\mathcal{C}$ *of* $\mathbb{R}^{t-d(\mathcal{A})-1} \setminus \Gamma_\sigma(\mathcal{A}, B)$ *a* signed reduced chamber. *We also call* $\mathcal{C}$ *an* outer *or* inner chamber, *according as* $\mathcal{C}$ *is unbounded or bounded.* $\diamond$

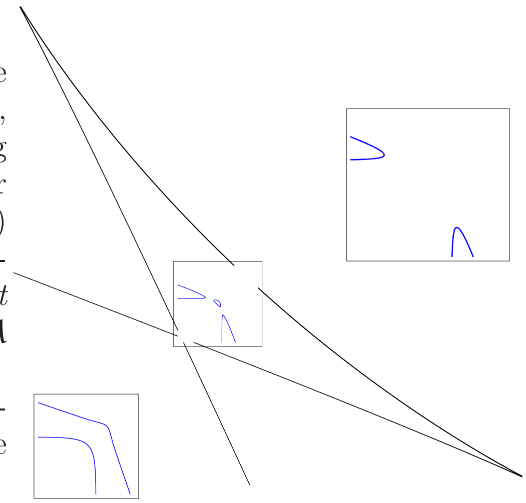
Example 2.6. *Continuing Example 2.4, there are* 16 *possible choices for* σ, *if we identify sign sequences with their negatives. Among these choices, there are* 11 σ *yielding* $\Gamma_\sigma(\mathcal{A}, B) = \emptyset$. *The remaining choices, along with their respective* $\Gamma_\sigma(\mathcal{A}, B)$ *are drawn below.* $\diamond$



Note that the curves drawn above are in fact unbounded, so the number of reduced signed chambers for the σ above, from left to right, is respectively 2, 2, 3, 2, and 2. (The tiny $\times$ in each illustration indicates the origin in $\mathbb{R}^2$.) In particular, only $\sigma = (1, -1, -1, 1, 1)$ yields an inner chamber. Note also that $\Gamma(\mathcal{A}, B)$ is always the union of all the $\Gamma_\sigma(\mathcal{A}, B)$.

Remark 2.7. *While the shape of the reduced signed chambers certainly depends on the choice of* B, *the hyperplane arrangement* $H_{\mathcal{A}}$ *and the number of signed chambers for any fixed* σ *are independent of* B. *In particular, working with the* $\Gamma_\sigma(\mathcal{A}, B)$ *(which have dimension*

$t-d(\mathcal{A})-2$) helps us visualize and work with the real part of $\Xi_{\mathcal{A}}$ (which has dimension $t-2$ and involves up to 2^{t-1} orthants). $\diamond$



It is also important to note that innner chambers are where the isotopy type of $Z_{\mathbb{R}}(g)$ becomes more subtle, and no longer approachable via classical patchworking [Vir84]. For instance, the 3 possible isotopy types for $Z_{\mathbb{R}}(g)$ in our last example (with $\sigma=(1,-1,-1,1,1)$) are drawn within the boxes to the right. Note in particular that the isotopy type with a compact oval is *not* obtainable by applying patchworking to the point set $\mathcal{A}$ in any obvious way.

We close this section with the key reason we introduced $\mathrm{Log}|\cdot|$ earlier: it makes studying the curvature of the image of $\xi_{\mathcal{A},B}$ much easier.

Theorem 2.8. *Suppose $\mathcal{A}\in\mathbb{R}^{n\times t}$ is non-defective. Then, at any point $[\ell]\in\mathbb{P}^{t-d(\mathcal{A})-2}_{\mathbb{R}}\setminus H_{\mathcal{A}}$ where $\xi_{\mathcal{A},B}$ is differentiable, we have that $v\in\mathbb{R}^{t-d(\mathcal{A})-1}\setminus\{\mathbf{O}\}$ is a normal vector to $\xi_{\mathcal{A},B}([\ell]) \Longleftrightarrow [v]=[\ell]$.* $\blacksquare$

Recall that the *Gauss map* from $\Gamma(\mathcal{A},B)$ to $\mathbb{P}^{t-d(\mathcal{A})-2}_{\mathbb{R}}$ is the map to $\mathbb{P}^{t-d(\mathcal{A})-2}_{\mathbb{R}}$ obtained by mapping a point of $\Gamma(\mathcal{A},B)$ to the direction of its normal line. A simple consequence of Theorem 2.8 is that the Gauss map in our setting is an injection with dense image. (An example is our illustration of $\Gamma(\mathcal{A},B)$ from Example 2.4.) The special case $\mathcal{A}\in\mathbb{Z}^{n\times t}$ of Theorem 2.8 was already observed by Kapranov in [Kap91, Thm. 2.1 (b)]. The proof of the more general Theorem 2.8 is in fact almost identically, since it ultimately reduces to elementary identities involving linear combinations of logarithms in linear forms in λ, where one merely needs $\mathcal{A}$ to be real.

3. Morse Theory, Completed Contours, and Roots at Infinity

Let us call $\mathcal{A}\in\mathbb{R}^{n\times t}$ *combinatorially simplicial* if and only if $\mathcal{A}\cap Q$ has cardinality $1+\dim Q$ for every proper face Q of $\mathrm{Conv}\{a_1,\ldots,a_t\}$. (The books [Grü03, Zie95] are excellent standard references on polytopes, their faces, and their normal vectors.) Note that $\mathrm{Conv}\{a_1,\ldots,a_t\}$ need *not* be a simplex for $\mathcal{A}$ to be combinatorially simplicial (consider, e.g., Example 2.4). We now state the main reason we care about reduced signed chambers.

Theorem 3.1. *Suppose $\mathcal{A}\in\mathbb{R}^{n\times t}$ is combinatorially simplicial, and g_1 and g_2 are each n-variate exponential t-sums with spectrum $\mathcal{A}$, real coefficients, and smooth real zero set. Suppose further that $\mathrm{sign}(c_{g_1})=\pm\mathrm{sign}(c_{g_2})$, and $(\mathrm{Log}|c_{g_1}|)B$ and $(\mathrm{Log}|c_{g_2}|)B$ lie in the same signed reduced discriminant chamber. Then $Z_{\mathbb{R}}(g_1)$ and $Z_{\mathbb{R}}(g_2)$ are ambiently isotopic in $\mathbb{R}^n$.*

We prove Theorem 3.1 after recalling one definition and one lemma on a variant of the classical *moment map* [Sma70, Sou70] from symplectic geometry. The special case $\mathcal{A}\in\mathbb{Z}^{n\times t}$ of Theorem 3.1, without the use of Log or B, is alluded to near the beginning of [GKZ94, Ch. 11, Sec. 5]. However, Theorem 3.1 is really just an instance of *Morse Theory* [Mil69, GM88], once one considers the manifolds defined by the fibers of the map

$Z_{\mathbb{R}}(g) \mapsto (\mathrm{Log}|c_g|)B$ along paths inside a fixed signed chamber. In particular, the assumption that $\mathcal{A}$ be combinatorially simplicial forces any topological change in $Z_{\mathbb{R}}(g)$ to arise solely from singularities of $Z_{\mathbb{R}}(g)$ in $\mathbb{R}^n$. When $\mathcal{A}$ is more general, topological changes in $Z_{\mathbb{R}}(g)$ can arise from pieces of $Z_{\mathbb{R}}(g)$ approaching infinity, with no singularity appearing in $\mathbb{R}^n$. So our chambers will eventually need to be cut into smaller pieces.

So let us now make the notion of roots at infinity rigorous.

Definition 3.2. *Let* $\mathrm{Int}(S)$ *denote the topological interior of any set* $S \subseteq \mathbb{R}^n$*, and let* $\mathrm{RelInt}(Q)$ *denote the relative interior of any* d*-dimensional polytope* $Q \subset \mathbb{R}^n$*, i.e.,* $Q \setminus R$ *where* R *is the union of all faces of* Q *of dimension strictly less than* d *(using* $\emptyset$ *as the only face of dimension* <0*). For any* $w \in \mathbb{R}^n \setminus \{\mathbf{O}\}$*, we define* $Q^w := \{x \in Q \mid x \cdot w = \min_{y \in Q} y \cdot w\}$ *to be the* face of Q with inner normal w. *Finally, for any real exponential sum* $g(y) = \sum_{j=1}^{t} c_j e^{a_j \cdot y}$*, let* $\mathrm{In}_w(g) := \sum_{a_j \in \mathrm{Conv}\{a_1,\ldots,a_t\}^w} c_j e^{a_j \cdot y}$ *be the* initial term summand of g with respect to the weight w. $\diamond$

Lemma 3.3. [LRW03, Lemma 14][3] *Given any* n*-dimensional convex compact polytope* $P \subset \mathbb{R}^n$*, there is a real analytic diffeomorphism* $\mu_P : \mathbb{R}^n \longrightarrow \mathrm{Int}(P)$*. Moreover, if* g *is an* n*-variate exponential* t*-sum with spectrum having convex hull* P *of dimension* n*, and* $w \in \mathbb{R}^n \setminus \{\mathbf{O}\}$*, then there is a real analytic diffeomorphism between* $Z_{\mathbb{R}}(\mathrm{In}_w(g)) \subset \mathbb{R}^n$ *and*

$$\left(\mathrm{RelInt}(P^w) \cap \overline{\mu_P(Z_{\mathbb{R}}(g))}\right) \times \mathbb{R}^{n-\dim P^w}.$$

In particular, $\mu_P(Z_{\mathbb{R}}(g))$ *has a limit point in* $\mathrm{RelInt}(P^w) \Longrightarrow \mathrm{In}_w(g)$ *has a root in* $\mathbb{R}^n$. ■

Note that the converse of Lemma 3.3 need not hold: A simple counter-example is $g(y) := (e^{2y_1} + e^{2y_2} - 1)^2 + (e^{y_1} - 1)^2$ and $w = (0, 1)$. In what follows, recall that n-manifolds (resp. n-manifolds with boundary) are defined by coordinate charts that are diffeomorphic to a (relatively) open subset of $\mathbb{R}^n$ (resp. $\mathbb{R}^{n-1} \times (\mathbb{R}_+ \cup \{0\})$) (see, e.g., [Hir94]). More generally, an *n-manifold with corners* is defined via charts in $(\mathbb{R}_+ \cup \{0\})^n$ instead (see, e.g., [GM88]). Manifolds with corners admit a natural Whitney stratification into (smooth, open) sub-manifolds without boundary. In fact, a by-product of Definition 3.2 is just such a stratification via the face lattice of P.

Proof of Theorem 3.1: Let $P := \mathrm{Conv}\{a_1, \ldots, a_t\}$, let $\mathcal{C}$ be the unique signed reduced chamber containing $(\mathrm{Log}|c_{g_1}|)B$ and $(\mathrm{Log}|c_{g_2}|)B$, let $\overline{\mathcal{C}}$ be the inverse image of $\mathcal{C}$ under $\mathrm{Log}|\cdot||_{\mathbb{P}^{t-1}_{\sigma}}$, let $\varphi = [\varphi_1 : \cdots : \varphi_t] : [0,1] \longrightarrow \overline{\mathcal{C}}$ be any analytic path connecting $[c_{g_1}]$ and $[c_{g_2}]$, and let

$$M := \left\{(y, s) \in \mathbb{R}^n \times [0,1] \;\middle|\; \sum_{j=1}^{t} \varphi_j(s) e^{a_j \cdot y} = 0\right\}.$$

Also let $p : M \longrightarrow [0,1]$ be the natural projection that forgets the first n coordinates of M. Observe then that the map between tangent spaces induced by p has deficient rank at $(y_0, s_0) \in M$ if and only if the partial derivatives $\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}$ of $g_s(y) := \sum_{j=1}^{t} \varphi_j(s) e^{a_j \cdot y}$ all vanish at (y_0, s_0). Since $Z_{\mathbb{R}}(g_s)$ is smooth for all $s \in [0,1]$ by construction, it follows that p has no critical points. Furthermore, letting $\overline{M}$ denote the Euclidean closure

$$\overline{\bigcup_{s \in [0,1]} \mu_P(Z_{\mathbb{R}}(g_s)) \times \{s\}},$$

it follows from Lemmata 2.1 and 3.3 that $\overline{M} \cap Q$ is a smooth hypersurface in $\mathrm{RelInt}(Q)$, for any face Q of $P \times [0,1]$. So then, $\overline{M}$ is a compact manifold with corners, $\overline{M} \cap (\mathrm{Int}(P) \times [0,1])$ is homeomorphic to M, and p extends naturally to a projection $\bar{p} : \overline{M} \longrightarrow [0,1]$ that also

[3] The statement in [LRW03] is for polynomials with real exponents but is equivalent, via the substitution $x_i = e^{y_i}$, to the statement here.

has no critical points. By the Regular Interval Theorem (see, e.g., [Hir94, Thm. 2.2, Pg. 153] for the case of manifolds with boundary), $\overline{M}$, $\overline{\mu_P(Z_{\mathbb{R}}(g_1))} \times [0,1]$, and $\overline{\mu_P(Z_{\mathbb{R}}(g_2))} \times [0,1]$ are homeomorphic to each other. In other words, the ends of $\overline{M}$ (which are $\mu_P(Z_{\mathbb{R}}(g_1)) \times \{0\}$ and $\mu_P(Z_{\mathbb{R}}(g_2)) \times \{0,1\}$) are ambiently isotopic in P. So $Z_{\mathbb{R}}(g_1)$ and $Z_{\mathbb{R}}(g_2)$ are ambiently isotopic in $\mathbb{R}^n$ and we are done. ■

To address arbitrary $\mathcal{A}$ we'll first need a little more terminology. This is because, for non-simplicial $\mathcal{A}$, our preceding argument can be obstructed by singularities "at infinity". So we will need to further partition our coefficient space to avoid these additional singularities. Fortunately, our development allows us to do this by simply by considering a few additional discriminant contours corresponding to "faces" of $\mathcal{A}$.

Definition 3.4. *Given any $\mathcal{A} \in \mathbb{R}^{n\times t}$ with distinct columns $a_1, \ldots, a_t$, and any inner normal $w \in \mathbb{R}^n$ to a face of* $\mathrm{Conv}\{a_1, \ldots, a_t\}$*, we let $[a_{j_1}, \ldots, a_{j_r}]$ be the sub-matrix of $\mathcal{A}$ corresponding to the set $\{a \in \mathcal{A} \mid a \cdot w = \min_{a' \in \mathcal{A}}\{a' \cdot w\}\}$. Setting $\mathcal{A}^w := [a_{j_1}, \ldots, a_{j_r}]$, we call $\mathcal{A}^w$ a* (proper) non-simplicial face *of $\mathcal{A}$ when $d(\mathcal{A}^w) \leq d(\mathcal{A}) - 1$ and $\mathcal{A}^w$ has at least $d(\mathcal{A}^w) + 2$ columns. Also let B^w be any matrix whose columns form a basis for the right nullspace of $\widehat{(\mathcal{A}^w)}$, and let $\pi_w : \mathbb{C}^t \longrightarrow \mathbb{C}^r$ be the natural coordinate projection map defined by $\pi_w(c_1, \ldots, c_t) := (c_{j_1}, \ldots, c_{j_r})$. When $\mathcal{A}$ is non-defective we then define the* completed *reduced signed contour,*

$$\widetilde{\Gamma}_\sigma(\mathcal{A}, B) \subset \mathbb{R}^{t-d(\mathcal{A})-1},$$

to be the union of $\Gamma_\sigma(\mathcal{A}, B)$ and

$$\bigcup_{\substack{\mathcal{A}^w \text{ a non-} \\ \text{simplicial} \\ \text{face of } \mathcal{A}}} \overline{\left\{ \pi_w^{-1}(\mathrm{Log}|\lambda(B^w)^\top|)\, B \;\middle|\; \mathrm{sign}\,(\lambda(B^w)^\top) = \pm\pi_w(\sigma) \ ,\ [\lambda] \in \mathbb{P}_{\mathbb{R}}^{t-d(\mathcal{A})-2} \setminus H_{\mathcal{A}} \right\}}.$$

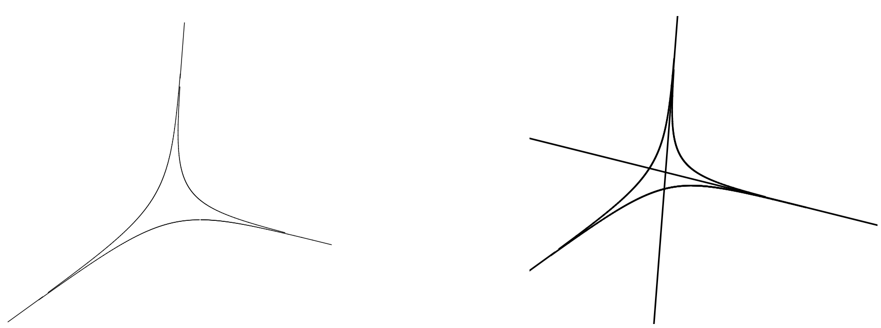
We call any unbounded connected component of $\mathbb{R}^{t-d(\mathcal{A})-1} \setminus \widetilde{\Gamma}_\sigma(\mathcal{A}, B)$ a refined *outer chamber. Finally, we call $\widetilde{\Gamma}(\mathcal{A}, B) := \bigcup\limits_{\sigma \in \{\pm 1\}^t} \widetilde{\Gamma}_\sigma(\mathcal{A}, B)$ a* completed reduced (unsigned) contour*.* ⋄

It is useful to remember that $(\mathrm{Log}|\psi_{\mathcal{A}}([\lambda], y)|)B = \xi_{A,B}([\lambda])$.

Example 3.5. *When $\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 2 \end{bmatrix}$ a short* `Matlab` *computation yields*

$$B \approx \begin{bmatrix} 0.4335 & -0.8035 & -0.0635 & 0.4018 & 0.0317 \\ 0.3127 & 0.2002 & -0.8256 & -0.1001 & 0.4128 \end{bmatrix}^\top$$

and the following corresponding reduced contour $\Gamma(\mathcal{A}, B)$ and completed reduced contour $\widetilde{\Gamma}(\mathcal{A}, B)$:



Note in particular that $\widetilde{\Gamma}(\mathcal{A}, B) = \Gamma(\mathcal{A}, B) \cup S_1 \cup S_2$ where S_1 and S_2 are lines that can be viewed as line bundles over points. These points are in fact $(\mathrm{Log}|\Xi_{\mathcal{A}_1}|)B$ and $(\mathrm{Log}|\Xi_{\mathcal{A}_2}|)B$ where $\mathcal{A}_1$ and $\mathcal{A}_2$ are the facets of $\mathcal{A}$ with respective outer normals $(-1, 0)$ and $(0, -1)$. ⋄

Call a non-simplicial face of $\mathcal{A}^w$ of $\mathcal{A}$ *minimal* if and only if $\mathcal{A}^w$ has no proper non-simplicial faces.

Proposition 3.6. *If $\mathcal{A}$ is non-defective and not combinatorially simplicial then $\mathcal{A}$ has at most $t-d(\mathcal{A})-1$ minimal non-simplicial faces $\mathcal{A}^w$.* ■

Proposition 3.7. *Suppose $t=n+3$, $\mathcal{A}$ has exactly 2 minimal non-simplicial facets, $d(\mathcal{A})=n$, and $B\in\mathbb{R}^{(n+3)\times 2}$ is any matrix whose columns form a basis for the right nullspace of $\widehat{\mathcal{A}}$. Then $\widetilde{\Gamma}(\mathcal{A},B)\setminus\Gamma(\mathcal{A},B)$ is a union of 2 lines.*

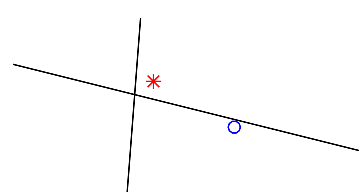
Proof: By assumption, $\mathcal{A}$ contains two *circuits*. In particular, this implies that there are two distinct subsets of indices $I_1, I_2\subset\{1,\ldots,n+3\}$, neither containing the other, such that $\dim\mathrm{Conv}\{a_i \mid i\in I_j\}=|I_j|-2$ for $j\in\{1,2\}$ (where we use $|S|$ for the cardinality of S). In particular, we obtain two linearly independent affine relations on the a_ℓ: one involving only $\{a_i \mid i\in I_1\}$ and the other involving only $\{a_i \mid i\in I_2\}$.

We can then take the columns of B to be the two preceding affine relations. To conclude, we simply observe that $\widetilde{\Gamma}(\mathcal{A},B)\setminus\Gamma(\mathcal{A},B)$ is simply the union of two contours, defined by the $\mathcal{A}$-discriminants of the subsets of $\mathcal{A}$ defined by I_1 and I_2. In particular, by Lemma 2.1, these contours are lines, and we are done. ■

Theorem 3.8. *Suppose $\mathcal{A}\in\mathbb{R}^{n\times t}$, and g_1 and g_2 are each n-variate exponential t-sums with spectrum $\mathcal{A}$, real coefficients, and smooth real zero set. Suppose further that $\sigma:=\mathrm{sign}(c_{g_1})=\pm\mathrm{sign}(c_{g_2})$, and $(\mathrm{Log}|c_{g_1}|)B$ and $(\mathrm{Log}|c_{g_2}|)B$ lie in the same connected component of $\mathbb{R}^{t-d(\mathcal{A})-1}\setminus\widetilde{\Gamma}_\sigma(\mathcal{A},B)$. Then $Z_\mathbb{R}(g_1)$ and $Z_\mathbb{R}(g_2)$ are ambiently isotopic in $\mathbb{R}^n$.*

Example 3.9. *Observe that the circle defined by $\left(u+\frac{1}{2}\right)^2+(v-2)^2=1$ intersects the positive orthant, while the circle defined by $\left(u+\frac{3}{2}\right)^2+\left(v-\frac{3}{2}\right)^2=1$ does not. Consider then $\mathcal{A}=\begin{bmatrix}0&1&0&2&0\\0&0&1&0&2\end{bmatrix}$ as in our last example, and let $g_1=\left(e^{y_1}+\frac{1}{2}\right)^2+\left(e^{y_2}-2\right)^2-1$ and $g_2=\left(e^{y_1}+\frac{3}{2}\right)^2+\left(e^{y_2}-\frac{3}{2}\right)^2-1$. Then g_1 and g_2 have spectrum $\mathcal{A}$,* $\mathrm{sign}(c_{g_1})=\mathrm{sign}(c_{g_2})=\sigma$ *with $\sigma=(1,1,-1,1,1)$, and $(\mathrm{Log}|c_{g_1}|)B$ and $(\mathrm{Log}|c_{g_2}|)B$ lie in the same reduced signed $\mathcal{A}$-discriminant chamber (since $\Gamma_\sigma(\mathcal{A},B)=\emptyset$ here). However, $Z_\mathbb{R}(g_1)$ consists of a single smooth arc, while $Z_\mathbb{R}(g_2)$ is empty. This is explained by the* completed *contour $\widetilde{\Gamma}_\sigma(\mathcal{A},B)$ consisting of two lines, and $(\mathrm{Log}|c_{g_1}|)B$ and $(\mathrm{Log}|c_{g_2}|)B$ lying in distinct* refined *chambers as shown, respectively via the symbols $\circ$ and $*$, below to the right.* ◇

Proof of Theorem 3.8: We follow exactly the same set-up as the proof of Theorem 3.1. However, although $\mathcal{A}$ may not be combinatorially simplicial, our use of $\widetilde{\Gamma}_\sigma(\mathcal{A},B)$ instead of $\Gamma_\sigma(\mathcal{A},B)$ guarantees that $\bar{p}$ is still a projection from $\overline{M}$ to $[0,1]$ without critical points. So we can again obtain an ambient isotopy via the Regular Interval Theorem as before. ■



Remark 3.10. *An alternative to our use of the moment map would have been to instead embed $\mathbb{R}^n$ into a suitable (compact)* quasi-fold *(see, e.g.,* [Pra01, BP02]*). Roughly, quasi-folds are to toric varieties what real exponents are to integral exponents.* ◇

We are now ready to prove our main results.

4. The Proof of Our Parametrization: Theorem 1.7

Let $c := (c_1, \ldots, c_t) \in (\mathbb{C} \setminus \{0\})^t$. Rewriting the equations defining $g(y) = \frac{dg}{dy_1} = \cdots = \frac{dg}{dy_n} = 0$, we see that g has a singularity at $y \in \mathbb{C}^n$ if and only if

$$\widehat{\mathcal{A}} \begin{bmatrix} c_1 e^{a_1 \cdot y} \\ \vdots \\ c_t e^{a_t \cdot y} \end{bmatrix} = \mathbf{O}.$$

The last equality is equivalent to $c \odot e^{y\mathcal{A}} = \lambda B^\top$ for some $[\lambda] \in \mathbb{P}_{\mathbb{C}}^{t-d(\mathcal{A})-1} \Big\backslash H_{\mathcal{A}}$, thanks to the definition of the right nullspace of $\widehat{\mathcal{A}}$. (Note that Proposition 2.2 implies that λ must be such that no coordinate of $\lambda B^\top$ vanishes.) Dividing the $i^{\underline{\text{th}}}$ coordinate of each side by $e^{a_j \cdot y}$, we thus obtain that the image $\psi_{\mathcal{A}}\left(\left(\mathbb{P}_{\mathbb{C}}^{t-d(\mathcal{A})-2} \Big\backslash H_{\mathcal{A}}\right) \times \mathbb{C}^n\right)$ is exactly $\Xi_{\mathcal{A}} \setminus \{c_1 \cdots c_t = 0\}$. So the last two sets have the same Euclidean closure. Note in particular that $\psi_{\mathcal{A}}$ is analytic on $\left(\mathbb{P}_{\mathbb{C}}^{t-d(\mathcal{A})-2} \Big\backslash H_{\mathcal{A}}\right) \times \mathbb{C}^n$, and $\left(\mathbb{P}_{\mathbb{C}}^{t-d(\mathcal{A})-2} \Big\backslash H_{\mathcal{A}}\right) \times \mathbb{C}^n$ is connected. (The latter assertion follows from the fact that complements of algebraic hypersurfaces in $\mathbb{C}^N$ are connected: See, e.g., [BCSS98, Pg. 196].) So $\psi_{\mathcal{A}}\left(\left(\mathbb{P}_{\mathbb{C}}^{t-d(\mathcal{A})-2} \Big\backslash H_{\mathcal{A}}\right) \times \mathbb{C}^n\right)$ is a connected analytic hypersurface and we thus obtain Assertion (1).

Let us now assume that all the entries of $\mathcal{A}$ are real. Since $\widehat{\mathcal{A}}B = \mathbf{O}$, we immediately obtain that the restriction of the map $x \mapsto xB$ to $\text{Log}|\Xi_{\mathcal{A}} \cap \mathbb{P}_\sigma^{t-1}|$ yields an $\mathbb{R}^n$-bundle over $Z := (\text{Log}\,|\Xi_{\mathcal{A}} \cap \mathbb{P}_\sigma^{t-1}|)\, B$. We will describe the subsets $\Gamma_\sigma(\mathcal{A})$ and Y_σ of Z as we prove Assertions (a) and (b).

Let $\Sigma \subset \mathbb{P}_{\mathbb{R}}^{t-d(\mathcal{A})-2}$ denote the singular locus of $\xi_{\mathcal{A},B}$. Since $\xi_{\mathcal{A},B}([\lambda]) = (\text{Log}|\psi_{\mathcal{A}}([\lambda, y])|)B$ for all λ and y, Theorem 2.8 implies that $\Gamma_\sigma(\mathcal{A}, B)$ is a dense open subset of Z, provided we prove that Σ is an algebraic set of positive codimension. This is indeed the case: Recalling that B has columns $b_1, \ldots, b_{t-d(\mathcal{A})-1}$, and rows $\beta_1, \ldots, \beta_t$, an elementary calculation shows that $\xi_{\mathcal{A},B}$ is singular at $[\lambda] \in \mathbb{P}_{\mathbb{C}}^{t-d(\mathcal{A})-1}$ if and only if some $(t - d(\mathcal{A}) - 2) \times (t - d(\mathcal{A}) - 1)$ sub-matrix of the $(t - d(\mathcal{A}) - 1) \times (t - d(\mathcal{A}) - 1)$ matrix

$$M(\lambda) := \begin{bmatrix} \left(b_1^\top \odot \left(\frac{1}{\lambda \cdot \beta_1^\top}, \ldots, \frac{1}{\lambda \cdot \beta_t^\top}\right)\right) B \\ \vdots \\ \left(b_{t-d(\mathcal{A})-1}^\top \odot \left(\frac{1}{\lambda \cdot \beta_1^\top}, \ldots, \frac{1}{\lambda \cdot \beta_t^\top}\right)\right) B \end{bmatrix}$$

has rank $< t - d(\mathcal{A}) - 2$, and this is clearly an algebraic condition (defined over $\mathbb{R}$ in fact). Letting $\Delta \subset \mathbb{P}_{\mathbb{R}}^{t-d(\mathcal{A})-1}$ be the open polyhedron consisting of all λ such that $\text{sign}(\lambda B^\top) = \pm\sigma$ we then obtain Assertion (a) (with $\Gamma_\sigma(\mathcal{A}) = \xi_{\mathcal{A},B}(\Delta \setminus \Sigma)$), save for the stated quantitative bound. To obtain this bound, observe that $\Delta \setminus \Sigma$ is the complement of a real algebraic hypersurface and thus has only finitely many connected components (see, e.g., [BCSS98, Ch. 16]), and $\xi_{\mathcal{A},B}$ is analytic on these connected components. We also obtain Assertion (b) (with Y_σ the codimension ≥ 2 part of Z), since any analytic set is a countable locally finite union of manifolds [Łoj91]. So we now only need to prove the quantitative bound from Assertion (a).

Let $R(\lambda) := (\det M(\lambda)) \left(\prod_{i=1}^t \lambda \cdot \beta_i^\top\right)^{t-d(\mathcal{A})-1}$. Clearly, $\Sigma \subseteq Z_{\mathbb{R}}(R)$. Also, since $\det M(\lambda)$ consists of a linear combination of products of terms of the form $\sum_{i=1}^t \frac{\gamma_i}{\beta_i \cdot \lambda}$, R is clearly homogeneous and has degree no greater than $(t-d(\mathcal{A})-1)(t-1)$. By the classical Oleinik-Petrovsky/Milnor-Thom bound (see, e.g., [BCSS98, Ch. 16, Pg. 308, Prop. 3]), $\Delta\setminus\Sigma$ has no more than $(t-d(\mathcal{A})-1)(t-1)(2(t-d(\mathcal{A})-1)(t-1)-1)^{2t-d-2}$ connected components. So we are done. ■

5. On the Number of Cells Determined by a Union of Convex Arcs: the Proof of Theorem 1.8

By definition, C is a union of $\ell+m$ smooth convex arcs $C_1,\ldots,C_{\ell+m}$ admitting a parametrization $\varphi:[0,1]\longrightarrow C$ such that $\varphi(0)=\varphi(1)$ and, for any $i\in\{1,\ldots,\ell+m\}$, there is a $\theta_i\in(0,1)$ with $(C_1\cup\cdots\cup C_i)\cap\varphi(s)=\emptyset$ for all $s>\theta_i$. In particular, we may assume $\varphi(0)$ is a cusp or a pole. Clearly then, a self-intersection for C corresponds to a pair (i,j) with $i<j$, $C_i\cap C_j\neq\emptyset$, $j\neq i+1$, and $(i,j)\neq(1,\ell+m)$. There are then clearly at most $\binom{\ell+m}{2}-(\ell+m)$ such pairs.

To prove the bound on the number of regions determined by the complement of C, we make a quick application of Euler's classical formula for the number, F, of regions on a sphere determined by the complement of a connected planar graph with exactly V vertices and E edgess [Bol02]: The triple (F,E,V) must satisfy $V-E+F=2$. For our union of curves C, we turn it into a graph simply by identifying cusps and self-intersections as vertices, the arcs C_i as edges, and adding an extra vertex at infinity (by embedding $\mathbb{R}^2$ into the sphere via stereographic projection) incident to any unbounded arc. In particular, our graph is connected, since any C_i is either incident to the vertex at infinity, a cusp connected to C_{i+1}, or a cusp connected to C_{i-1} (letting $C_0:=C_{\ell+m}$ and $C_{\ell+m+1}=C_1$).

For our case at hand, V is at least $\ell+1$ thanks to the vertex at infinity. Also, E is at most $\ell+\frac{(\ell+m)(\ell+m-1)}{2}-(\ell+m)$ (the sum of the number of cusps and the maximal number of self-intersections). This is because each edge is incident to exactly 2 vertices, and the edges can be ordered circularly by our parametrization φ. So then, Euler's Formula implies that $F=2+E-V\leq 2+\ell+\frac{(\ell+m)(\ell+m-1)}{2}-(\ell+m)-(\ell+1)=\frac{(\ell+m)(\ell+m-1)}{2}-(\ell+m)+1$, and we are done. ■

6. Counting Isotopy Types: Proving Theorem 1.1 and Lemma 1.2

The final key idea toward proving Theorem 1.1 is to show that the convex arcs making up the reduced discriminant have few singularities, thus enabling a further reduction to counting cells in an arrangement of few lines. Toward this end, we'll need the following convenient characterization of the singularities of a reduced contour when $t=n+3$.

Lemma 6.1. *If $\mathcal{A}\in\mathbb{R}^{n\times(n+3)}$ is non-defective then any reduced $\mathcal{A}$-discriminant contour has no more than n cusps.*

Proof: Considering $\xi_{\mathcal{A}}([\lambda_1:\lambda_2])=(\xi_1([\lambda_1:\lambda_2]),\xi_1([\lambda_1:\lambda_2]))$ and dehomogenizing by setting $(\lambda_1,\lambda_2)=(1,\lambda)$, we can detect cusps by setting $\frac{\partial\psi_1}{\partial\lambda}=\frac{\partial\psi_2}{\partial\lambda}=0$. One then obtains a pair of equations of the form $\sum_{i=1}^{n+3}\frac{u_i}{\beta_{i,1}+\beta_{i,2}\lambda}=0$, with the sum on the left-hand side *not* identically 0 (thanks to Theorem 2.8). In fact, if we can combine terms with denominators identical up to scaling, to obtain a pair of equations of the form $\sum_{i=1}^{r}\frac{u'_i}{\beta_{i,1}+\beta_{i,2}\lambda}=0$ with $r\in\{3,\ldots,n+3\}$, each u'_i nonzero, and the points $[\beta_{1,1}:\beta_{1,2}],\ldots,[\beta_{r,1}:\beta_{r,2}]$ distinct in $\mathbb{P}^1_{\mathbb{R}}$. (That $r\geq 3$ follows from the fact that the rows of B must sum to $[0,0]$.) Clearing denominators, we then obtain a pair of *polynomial* equations, each involving a nonzero polynomials of degree $\leq r\leq n+2$.

One can then check that the coefficients of λ^{n-1} and λ^{n-2} in these polynomials are exactly 0, and thus we in fact obtain a pair of (univariate) polynomial equations involving polynomials of degree $\leq n$. So we clearly obtain that no more than n points of the form $[\lambda_1 : \lambda_2] \in \mathbb{P}^1_{\mathbb{R}}$ can yield a cusp. ■

Combining Lemma 6.1 and Theorem 1.8 we then obtain the following:

Corollary 6.2. *For any non-defective $\mathcal{A} \in \mathbb{R}^{n\times(n+3)}$, the complement of the reduced $\mathcal{A}$-discriminant contour consists of no more than $2n^2+3n+1$ connected components.*

Proof: Our map $\xi_{\mathcal{A},B}$ clearly has at most $n+3$ poles. By Lemma 6.1, $\xi_{\mathcal{A},B}$ has at most n cusps. So Theorem 1.8 implies that the complement of the reduced $\mathcal{A}$-discriminant contour has no more than $\frac{(n+n+3)(n+n+3-1)}{2} - (n+n+3)+1 = 2n^2+3n+1$ connected components. ■

The final fact we'll need before proving Theorem 1.1 is a refinement of Theorem 1.1 involving $\mathcal{A} \in \mathbb{R}^{n\times(n+2)}$:

Lemma 6.3. *Assume $\{a_1,\ldots,a_{n+2}\} \subset \mathbb{R}^n$ does not lie in any affine hyperplane and assume $\mathcal{A} = [a_1,\ldots,a_{n+2}]$ and $B \in \mathbb{R}^{(n+2)\times 2}$ satisfy the hypotheses of Definition 1.5. Then for any $\sigma \in \{\pm 1\}^t$, the image $\mathrm{Log}|\mathbb{P}^{n+1}_\sigma \cap \Xi_{\mathcal{A}}|$ is either empty or a fixed affine hyperplane depending only on $\mathcal{A}$.* ■

The proof of Lemma 6.3 is simply a stream-lining of the proof of Theorem 1.7, based on the fact that the reduced $\mathcal{A}$-discriminant contour is just a point when $\mathcal{A} \in \mathbb{R}^{n\times(n+2)}$ has $\mathrm{rank}\left(\widehat{\mathcal{A}}\right) = n+1$.

Proof of Theorem 1.1: The special case where $\mathrm{Conv}\{a_1,\ldots,a_{n+3}\}$ is combinatorially simplicial follows from Corollary 6.2 and Theorem 3.1.

To address the general case, first assume $\mathcal{A}$ is defective. So $\mathrm{codim}\,\Xi_{\mathcal{A}} \geq 2$, and thus (applying the Regular Interval Theorem again, as in Theorem 3.1) $\mathbb{P}^{t-1}_{\mathbb{R}} \setminus \Xi_{\mathcal{A}}$ is path-connected, thereby implying that there is only 1 isotopy type for $Z_{\mathbb{R}}(g)$. So we may assume $\mathcal{A}$ is non-defective.

By Proposition 3.6, we may also assume $\mathrm{Conv}\{a_1,\ldots,a_{n+3}\}$ has exactly two facets containing exactly $n+1$ distinct points of $\{a_1,\ldots,a_{n+3}\}$. Proposition 3.7 then tells us that the resulting signed reduced contour consists of 1 closed locally convex arc C (with $\leq n$ cusps and $\leq n+3$ poles) and 2 lines we'll call L_1 and L_2. Considering the connected components of $\mathbb{R}^2 \setminus C$, we see that $\mathbb{R}^2 \setminus (C \cup L_1)$ has no more than $2(n+(n+3))+1 = 4n+7$ new connected components. This is because C consists of at most $n+(n+3) = 2n+3$ convex sub-arcs, L_1 can intersect a convex curve in at most 2 points, and each new connected component is incident to a connected component of $L_1 \setminus C$. Similarly, considering L_1 and L_2 intersect in at most one point, we see that $\mathbb{R}^2 \setminus (C \cup L_1 \cup L_2)$ has no more than $2(n+(n+3))+1+1 = 4n+8$ new connected components in addition to those of $\mathbb{R}^2 \setminus (C \cup L_1)$. So we see that $\mathbb{R}^2 \setminus (C \cup L_1 \cup L_2)$ has no more than

$$2n^2+3n+1+(4n+7)+(4n+8) = 2n^2+11n+16$$

connected components where $Z_{\mathbb{R}}(g)$ has constant topology for a fixed signed vector. So we are done. ■

6.1. **Proof of an Isotopy Lower Bound: Lemma 1.2.** Substituting $x_i = e^{y_i}$, we then see that the lower bound for $\mathcal{A} \in \mathbb{Z}^{1\times t}$ can be attained by counting the possible numbers of positive roots of a univariate t-nomial. Via a classic refinement of Descartes' Rule (see, e.g., [SL54, Gra99]), a t-nomial can attain k roots for any $k\in\{0,1,\ldots,t-1\}$ and we thus obtain the exact count of t for the number of isotopy types of g.

The lower bound for $\mathcal{A}\in\mathbb{Z}^{2\times t}$ follows via Viro's Patchworking. In particular, in [OK03], it is proved that the number of isotopy types for a real projective hypersurface defined by a homogeneous $(d+1)$-variate polynomial of degree r is $2^{\Omega(r^d)}$. This construction in fact yields asymptotically the same number of isotopy types for a hypersurface in $\mathbb{R}^d_+$ defined by a d-variate polynomial of degree r, since we can use translation of the variables to move any ovals into the positive orthant. The number of monomial terms of such a polynomial is $\frac{(r+d)\cdots(r+1)}{d!}=\Theta(r^d)$ for fixed d and increasing r. So this family of polynomials evinces $2^{\Omega(t)}$ distinct isotopy types for positive zero sets. Substituting exponentials for the variables, we are done. ■

Remark 6.4. *We have so far not addressed the conceptually simpler question of bounding the number of connected components of $Z_{\mathbb{R}}(g)$. In particular, we emphasize that the number of isotopy types can be far larger than the maximal number of connected components. For instance, in our preceding proof, we saw already that degree r curves have $2^{\Omega(r^2)}$ isotopy types for their positive zero sets, while the maximal number of connected components of such a zero set is easily seen to be $O(r^2)$ by translating ovals and an application of Harnack's classical estimate. (*[Vir08] *contains an elegant discussion of Harnack's estimate.)* ⋄

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