Attitude and Position Estimation from Vector Observations *

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Abstract

This paper introduces three novel methods to evaluate attitude and position from vector observations using a vision-based technology camera. The first approach, called *Linear Algebra Resection Approach* (LARA), solves for attitude and position simultaneously and can be used in the *lost-in-space* case, when no approximate solution is available. The solution is shown to be the left eigenvector associated with the minimum singular value of a rectangular data matrix. The second and third approaches, called *Attitude Free Approaches* (AFA), recast the problem into a nonlinear system of equations in terms of the unknown position only. Two different methods are proposed to solve this nonlinear set of equations. The First AFA (FAFA) uses a least-square Newton-Raphson iterative procedure and is particularly suitable for the recursive case, while the Second AFA (SAFA) uses the *toric resultant* to eliminate two variables from the attitude-free system of nonlinear polynomial equations and a discretization of the Cauchy integral theorem to quickly isolate the solution. SAFA can be used either in the *lost-in-space* or in the recursive cases. Final numerical tests quantify the robustness of these methods in the case of measurements affected by noise.

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Introduction

Proximity navigation as well as landing, docking, in flight re-fuelling, and many other tasks, can be well accomplished using the VISion-based NAVigation (VISNAV) technology[1, 2, 3, 4, 5, 6, 7, 8]. This technology uses a new sensing concept: an analog sensor that can near-simultaneously measure precisely the direction toward many discrete optical targets. The new sensor has an analog detector in the focal plane with a rise time of a few microseconds. Accuracies of better than one part in 2,000 of the field of view can be obtained. As a consequence, we can structure the light emitted by a set of actively controlled light emitting diodes (LEDs), in a fashion analogous to radar modulation/de-modulation, enabling easy rejection of ambient optical energy. The LEDs serve as actively commanded (cooperative) beacon targets; the line of sight toward these targets can be measured precisely. This enables six Degree Of Freedom (DOF) navigation with essentially zero image processing and no need for pattern recognition, as well as adaptively optimized signal to noise ratio for each measurement. A unique point-by-point (i.e., deterministic or geometric) 6-DOF solution must have a minimum of 4 beacons to obtain a unique solution of the relative navigation problem[1]. However, it is well known that by adding dynamic models in filter design can reduce the number of required measurements at each time instant (i.e., beacons). Furthermore, as the angular separation between beacons becomes smaller, which will occur for spacecraft borne sensors and beacons at greater distances between them, the accuracy of the deterministic solution degrades[1].

This paper presents two different mathematical approach to evaluate attitude and position from vector observation, as sketched below. The first approach tackles a sort of *Generalized Wahba Problem*[9] where, in addition to the attitude estimation, the evaluation of the position is also required. The second approach is based on the fact that our problem can be reduced to three quadratic equations in three unknowns, and is thus amenable to efficient recent techniques from computational algebraic geometry.

The first approach consists of a novel *Linear Algebra Resection Approach* (LARA) that provides a rigorous linearization of the geometric 6-DOF position and orientation estimation problem, if six or more beacons are available. Our analytic and numerical work establishes this important new algorithm for 6-DOF proximity navigation, geometric camera resection, and 6-DOF navigation. Both the classical batch least squares and the Extended Kalman Filter estimation approaches require iterations based upon a nonlinear measurement model (the co-linearity equations). We have recently discovered a new method for estimating the full 6-DOF navigation solution from vector line of sight measurements. This solution is of fundamental significance in its own right, or as a preliminary estimate to initiate other estimation algorithms. The new method is non-iterative and is a rigorous linear algebra-based technique that constitutes a promising alternative approach. This LARA technique provides the solution (position and orientation) in a single-point geometric fashion, analogous to the now-classical solutions of the classical Wahba problem[9]. Actually the classical

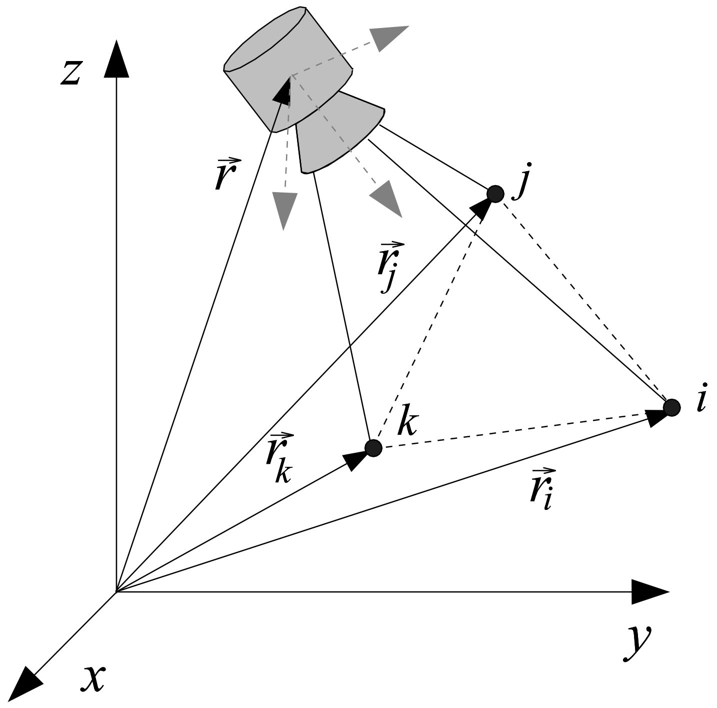

Figure 1: VISNAV geometrical definitions

Wahba problem is equivalent to the special case of our proximity camera resection problem where the observed targets are at infinite range from the sensor. The LARA solution is of basic theoretical and practical significance, and will likely be utilized as a starting algorithm for the nonlinear filter algorithms suitable for recursive navigation. Since the relative orbital and attitude models for spacecraft in formation often strive for high accuracies, the present formulation can significantly improve the accuracy at greater distances, which is also be studied.

The second approach is based on two deeper mathematical ideas and greatly simplifies the underlying nonlinearity of the problem. In particular, the *resultant* is a classical tool for eliminating variables from systems of (possibly nonlinear) polynomial equations, and we employ a modern variant which efficiently eliminates several variables at once. To summarize, there is a simple formulation of our problem in terms of just the three position variables, and via a multivariate resultant calculation, we reduce our problem to a single polynomial of degree 8 in just one position variable. Once we solve the latter univariate polynomial (a problem for which certifiably accurate algorithms with mathematically guaranteed convergence abound [10, 11]), the remaining two variables be solved for almost trivially. The particular approach we advocate for solving the resulting degree 8 polynomial is a hybrid algorithm that combines the classical Cauchy residue theorem [12] with Newton's method and is particularly efficient. Once we find the position variables, we can then adopt, say, the ESOQ-2 approach[13] which is one of the fastest among the optimal attitude determination algorithms. We thus present two options

1. use LARA as a simple preprocessor to other standard nonlinear equation solving algorithms, or

2. use a resultant calculation to reduce the whole problem to one variable and then isolate the real roots by some clever residue calculations.

We hope to make a comparison of these two approaches in a follow-up paper.

Linear Algebra Resection Approach

Let $\vec{R} = \{X \vdots Y \vdots Z\}^{\mathrm{T}}$ be the camera object space position vector and C be the direction cosine matrix representing the orientation of the camera. These parameters constitute the unknowns of our problem. Let the given i-th beacon be at position described by the vector $\vec{R}_i = \{X_i \vdots Y_i \vdots Z_i\}^{\mathrm{T}}$ which is observed in the camera focal plane by the coordinates (x_i, y_i). For the sake of the current discussion, let the positions $\vec{R}_i$ be known. Let f and (x_0, y_0) be the focal length and the optical axis offset of the camera, respectively. Provided that the pin-hole model holds for the camera, the co-linearity equations assume the form

$$\begin{cases} x_i = x_0 - f\left[\dfrac{C_{11}(X_i - X) + C_{12}(Y_i - Y) + C_{13}(Z_i - Z)}{C_{31}(X_i - X) + C_{32}(Y_i - Y) + C_{33}(Z_i - Z)}\right] \\ y_i = y_0 - f\left[\dfrac{C_{21}(X_i - X) + C_{22}(Y_i - Y) + C_{23}(Z_i - Z)}{C_{31}(X_i - X) + C_{32}(Y_i - Y) + C_{33}(Z_i - Z)}\right] \end{cases} \tag{1}$$

Alternatively and equivalently, we can write the corresponding vector form of the co-linearity equations, relating reference vector in the camera body coordinate frame to the corresponding vector in the object space frame, as the following relationship

$$p_i \begin{Bmatrix} x_0 - x_i \\ y_0 - y_i \\ f \end{Bmatrix} + \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{Bmatrix} X_i - X \\ Y_i - Y \\ Z_i - Z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{2}$$

where

$$p_i = \sqrt{\frac{(X_i - X)^2 + (Y_i - Y)^2 + (Z_i - Z)^2}{(x_0 - x_i)^2 + (y_0 - y_i)^2 + f^2}} \tag{3}$$

In lieu of expressing the camera position in the object space coordinates (X, Y, Z), we can alternatively choose to parameterize the unknown camera position in terms of camera space coordinates

$$B = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} \tag{4}$$

Consequently, Eq. (2) can be re-written in the alternative form

$$p_i \left\{ \begin{array}{c} x_0 - x_i \\ y_0 - y_i \\ f \end{array} \right\} - \left\{ \begin{array}{c} x \\ y \\ z \end{array} \right\} + \left[\begin{array}{ccc} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{array} \right] \left\{ \begin{array}{c} X_i \\ Y_i \\ Z_i \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right\} \tag{5}$$

Setting

$$\eta_i = \left\{ \begin{array}{c} x_0 - x_i \\ y_0 - y_i \\ f \end{array} \right\} \quad \text{and} \quad \xi_i = \left[\begin{array}{ccccccccc} X_i & Y_i & Z_i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & X_i & Y_i & Z_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & X_i & Y_i & Z_i \end{array} \right] \tag{6}$$

and

$$C_v = \{ C_{11} \vdots C_{12} \vdots C_{13} \vdots C_{21} \vdots C_{22} \vdots C_{23} \vdots C_{31} \vdots C_{32} \vdots C_{33} \}^{\mathrm{T}} \tag{7}$$

then, Eq. (5) suggests re-collecting the three equations, for each of n measurement sets, as a homogeneous linear form

$$[\ \eta_i \vdots - I_3 \vdots \xi_i\ ] \left\{ \begin{array}{c} p_i \\ B \\ C_v \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \right\} \tag{8}$$

The above provides n sets of 3 equations, can alternatively be written as the single homogeneous matrix equation

$$\left[\begin{array}{cccccc} \eta_1 & 0 & \cdots & 0 & -I_3 & \xi_1 \\ 0 & \eta_2 & \cdots & 0 & -I_3 & \xi_2 \\ 0 & \vdots & \ddots & \vdots & & \\ 0 & 0 & \cdots & \eta_n & -I_3 & \xi_n \end{array} \right] \left\{ \begin{array}{c} p_1 \\ p_2 \\ \vdots \\ p_n \\ B \\ C_v \end{array} \right\} = D\,X = \left\{ \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \right\} \tag{9}$$

Thus it appears that the $(12+n)$-long solution vector (redundant, by virtue of the fact that C contains only 3 independent degrees of freedom and the $(n+3)$ position coordinates $\{r_i,\, x_c,\, y_c,\, z_c\,\}$ also contain only 3 independent degrees of freedom for a total of 6 as is well known) lies in the null space of the $(3n) \times (12+n)$ coefficient matrix. Note that this matrix depends only on the object and image space coordinates of the measured points, as well as the camera calibration parameters $\{\, x_0,\, y_0,\, f\,\}$. It is evident that if this works, going to a heavily redundant coordinate description allows us to rigorously linearize the co-linearity equations - we can impose the constraints after the fact and thereby avoid the iteration and the issues associated with obtaining best estimates with potential problems associated with domain of convergence, etc. Also, because of the linear system - we should be able to find all solutions at once, with the possible headache of requiring more than the minimum number of measurements (theoretically, we require at least 3, but it is well known that at least 4 are required to avoid the issue of spurious solutions that satisfy the vector measurements). To obtain a unique solution by null space analysis of Eq. (9), we must have maximum rank $(11+n)$, so n must be greater or equal to 6, apparently.

However, the solution provided by Eq. (9) (independently from the sign and the modulus of X) is not anymore unique when the measurements are affected by noise, no matter how small this noise is. The problem arises because, in the presence of noise, the zero singular value (the minimum singular value) starts to differ from zero. This means, however, that the solution of Eq. (9) can be seen as the the left eigenvector associated with the minimum singular value of the D matrix. Now, under the hypothesis that the variation of the minimum singular value is so small that the solution is still associated with it, then our mathematical problem can be re-casted and solved through SVD. The SVD of D can be written as

$$D = R\,\Sigma\,L^{\mathrm{T}} \tag{10}$$

where Σ, R, and L, represent the singular value matrix, and the right and left eigenvector matrices, respectively. In particular the left eigenvector matrix can be computed as

$$(D^{\mathrm{T}}D)\,L = L\,\Sigma \tag{11}$$

Therefore, the SVD solution X_{sol} implies

$$(D^{\mathrm{T}}D)\,X_{sol} = \sigma_{\min}\,X_{sol} \tag{12}$$

Once X_{sol} has been computed, then the normalization to 1 of any row/column of the C matrix implies the correct re-normalization of the X_{sol} solution. As for the $\pm$ sign ambiguity for X_{sol}, since we do know that the attitude matrix must be proper, $\det(C) = +1$, then if we obtain $\det(C) = -1$, then we do need to change the sign of the solution X_{sol}.

First Attitude Free Approach

Here we consider the problem to identify the position vector $\vec{r}$ and the attitude C of a camera that measures the set of n directions $\hat{b}_k$, $k = 1, \ldots, n$, of beacons located at the positions $\vec{r}_k$ in the inertial reference frame. The camera-to-beacon vector can be expressed in the two coordinate system as

$$C\,\frac{1}{m_i}\,(\vec{r}_i - \vec{r}\,) = \hat{b}_i \qquad \text{where} \qquad m_i = \sqrt{(\vec{r}_i - \vec{r}\,)^{\mathrm{T}}(\vec{r}_i - \vec{r}\,)} \tag{13}$$

For the two beacons, "i" and "j", we can write

$$C\,(\vec{r}_i - \vec{r}\,) = m_i\,\hat{b}_i \qquad \text{and} \qquad C\,(\vec{r}_j - \vec{r}\,) = m_j\,\hat{b}_j \tag{14}$$

Subtracting member at member we obtain

$$C\,(\,\vec{r}_i - \vec{r}_j\,) = m_i\,\hat{b}_i - m_j\,\hat{b}_j \tag{15}$$

where the left side is a function of the camera orientation only, and the right side of the camera position only. Let us consider now the three beacons i, j, and k. In this case we can write Eq. (15) for each pair, and compact these three equations as follows

$$C\,[\vec{r}_i - \vec{r}_j \,\vdots\, \vec{r}_j - \vec{r}_k \,\vdots\, \vec{r}_k - \vec{r}_i\,] = [\,m_i\hat{b}_i - m_j\hat{b}_j \,\vdots\, m_j\hat{b}_j - m_k\hat{b}_k \,\vdots\, m_k\hat{b}_k - m_i\hat{b}_i\,] \tag{16}$$

Equation (16) can be written in the matrix form

$$C\,R = B(\vec{r}) \tag{17}$$

where

$$\begin{cases} R := [\,\vec{r}_i - \vec{r}_j \,\vdots\, \vec{r}_j - \vec{r}_k \,\vdots\, \vec{r}_k - \vec{r}_i\,] \\ B := [\,m_i\hat{b}_i - m_j\hat{b}_j \,\vdots\, m_j\hat{b}_j - m_k\hat{b}_k \,\vdots\, m_k\hat{b}_k - m_i\hat{b}_i\,] \end{cases} \tag{18}$$

By transposing Eq. (17) we obtain

$$R^{\mathrm{T}}C^{\mathrm{T}} = B^{\mathrm{T}}(\vec{r}\,) \tag{19}$$

premultiplying Eq. (17) by Eq. (19), since C is orthogonal, we obtain an interesting equality between two symmetric matrices

$$B^{\mathrm{T}}(\vec{r}\,)\,B(\vec{r}\,) = R^{\mathrm{T}}R \tag{20}$$

By equating the diagonal terms, Eq. (20) yields to the set of equations

$$\begin{cases} m_i^2 + m_j^2 - 2\,m_i\,m_j\,c_{ij} = d_{ij}^2 \\ m_j^2 + m_k^2 - 2\,m_j\,m_k\,c_{jk} = d_{ki}^2 \\ m_k^2 + m_i^2 - 2\,m_k\,m_i\,c_{ki} = d_{ki}^2 \end{cases} \tag{21}$$

while by equating the off-diagonal terms we obtain

$$\begin{cases} m_i m_j\,c_{ij} - m_k m_i\,c_{ki} - m_j^2 + m_j m_k\,c_{jk} = d_{ij}^2 \\ m_j m_k\,c_{jk} - m_i m_j\,c_{ij} - m_k^2 + m_k m_i\,c_{ki} = d_{jk}^2 \\ m_k m_i\,c_{ki} - m_j m_k\,c_{jk} - m_i^2 + m_i m_j\,c_{ij} = d_{ki}^2 \end{cases} \tag{22}$$

where

$$\begin{cases} c_{ij} = \hat{b}_i^{\mathrm{T}}\hat{b}_j \\ c_{jk} = \hat{b}_k^{\mathrm{T}}\hat{b}_j \\ c_{ki} = \hat{b}_i^{\mathrm{T}}\hat{b}_k \end{cases} \quad \text{and} \quad \begin{cases} d_{ij}^2 = (\vec{r}_i - \vec{r}_j)^{\mathrm{T}}(\vec{r}_i - \vec{r}_j) \\ d_{jk}^2 = (\vec{r}_j - \vec{r}_k)^{\mathrm{T}}(\vec{r}_j - \vec{r}_k) \\ d_{ki}^2 = (\vec{r}_k - \vec{r}_i)^{\mathrm{T}}(\vec{r}_k - \vec{r}_i) \end{cases} \tag{23}$$

Equations (21) and (22) represent two set of three nonlinear equations in the three unknowns identifying the coordinate of the camera. These two set of equations are, actually, equivalent to each other[¶]. Both equations are nothing more than the *law of cosines* in spherical trigonometry. However, Eq. (21) is applied to the triangle pij,

[¶]In fact the first of Eq. (22) can be obtained as half of the difference between the third of Eq. (21) and the first twos.

where p is the position of the observer, while Eq. (22) represents an interesting way to apply this law to the triangle ijk.

In the following, two different approaches are presented to solve these two set of equations. The first one is an iterative procedure based on Newton-Raphson approximation of the derivative and can be applied in the *recursive* approach, that is, when attitude and position information[‖] is available with an acceptable degree of accuracy. For instance, this approach can be adopted when the problem is solved with a given time frequency and the relative attitude and position variations are so small that linearization are good approximation. This approach is developed in the next two subsections.

Iterative solution using Eq. (21)

The diagonal elements of Eq. (20) implies the relationship

$$F_{ij} = m_i^2 + m_j^2 - 2\, m_i\, m_j\, c_{ij} - d_{ij}^2 = 0 \tag{24}$$

The expressions of the camera-to-beacon distances m_i are

$$m_i = \sqrt{(X - X_i)^2 + (Y - Y_i)^2 + (Z - Z_i)^2} \tag{25}$$

In order to apply the Newton-Raphson iterative technique to find the solution, the three derivatives (element of the Jacobian) of this function can be evaluated. For the x derivative we have

$$\frac{\partial F_{ij}}{\partial X} = 2\, m_i \frac{\partial m_i}{\partial X} + 2\, m_j \frac{\partial m_j}{\partial X} - 2 \left(m_i \frac{\partial m_j}{\partial X} + \frac{\partial m_i}{\partial X}\, m_j \right) c_{ij} \tag{26}$$

that can also be written in the more compact form

$$\frac{\partial F_{ij}}{\partial X} = 2 \left[(\, m_i - m_j\, c_{ij}\,) \frac{\partial m_i}{\partial X} + (\, m_j - m_i\, c_{ij}\,) \frac{\partial m_j}{\partial X} \right] \tag{27}$$

and where the expressions of the two derivatives are

$$\frac{\partial m_i}{\partial X} = \frac{X - X_i}{m_i} \qquad \text{and} \qquad \frac{\partial m_j}{\partial X} = \frac{X - X_j}{m_j} \tag{28}$$

Substituting Eq. (28) into Eq. (27) we obtain

$$\frac{\partial F_{ij}}{\partial X} = 2 \left[\left(1 - \frac{m_j}{m_i}\, c_{ij} \right) (X - X_i) + \left(1 - \frac{m_i}{m_j}\, c_{ij} \right) (X - X_j) \right] \tag{29}$$

Analogously, for the other two derivatives we obtain

$$\frac{\partial F_{ij}}{\partial Y} = 2 \left[\left(1 - \frac{m_j}{m_i}\, c_{ij} \right) (Y - Y_i) + \left(1 - \frac{m_i}{m_j}\, c_{ij} \right) (Y - Y_j) \right] \tag{30}$$

[‖]Actually, only an approximate knowledge of the position is required.

and

$$\frac{\partial F_{ij}}{\partial Z} = 2\left[\left(1 - \frac{m_j}{m_i}\,c_{ij}\right)(Z - Z_i) + \left(1 - \frac{m_i}{m_j}\,c_{ij}\right)(Z - Z_j)\right] \tag{31}$$

Equations (29) through (31) can be written down for all the possible i-j observed beacons pairs. If n is the number of observed beacons, then a set of $N = \dfrac{n(n-1)}{2}$ beacon pairs are available. Therefore the solving system is made of system of equations

$$J\,\delta\vec{r} = \begin{bmatrix} \frac{\partial F_{12}}{\partial X} & \frac{\partial F_{12}}{\partial Y} & \frac{\partial F_{12}}{\partial Z} \\ \vdots & \vdots & \vdots \\ \frac{\partial F_{(n-1)n}}{\partial X} & \frac{\partial F_{(n-1)n}}{\partial Y} & \frac{\partial F_{(n-1)n}}{\partial Z} \end{bmatrix} \left\{ \begin{array}{c} \delta X \\ \delta Y \\ \delta Z \end{array} \right\} = -\left\{ \begin{array}{c} F_{12} \\ \vdots \\ F_{(n-1)n} \end{array} \right\} = -\vec{F} \tag{32}$$

which allows us to write the least square solution for the position correction

$$\delta\vec{r} = -(\,J^{\mathrm{T}}\,J\,)^{-1}\,J^{\mathrm{T}}\,\vec{F} \tag{33}$$

that allows us to implement the iterative procedure (Newton-Raphson)

$$\vec{r}_{\ell+1} = \vec{r}_\ell + \delta\vec{r}_\ell \tag{34}$$

Once the convergence is achieved, then the attitude is simply computed with an algorithm fully complying with Wahba's optimality criterion. In doing this, the Wahba loss function, indicates whether the iterative procedure converged onto the correct solution (global minimum) or onto local one (relative minimum). In fact, if the camera position is wrong, then the angular structures of the observations and of the reference directions badly fit. Therefore, the loss function can be used as an indicator of the position correctness! A potential pit fall of this logic lies in the possibility that measurement error makes it impossible to distinguish between local solutions. This also depends on the initial estimated solution accuracy. However, extensive numerical tests, performed under a real assumption of the error contribution and of the initial estimated solution accuracy, never experienced failure generated by converging onto false local minima. Actually, even with very bad initial position estimate the number of failure are very, very, rare.

Once the camera position has been estimated, then the estimation of the camera attitude is accomplished by using ESOQ-2, the fastest existing optimal attitude estimation algorithm.[13, 14]

Iterative solution using Eq. (22)

Let us identify the last of Eqs. (22) as F_{ijk}

$$F_{ijk} = m_k m_i c_{ki} - m_j m_k c_{jk} - m_i^2 + m_i m_j c_{ij} - (\vec{r}_k - \vec{r}_i)^{\mathrm{T}}(\vec{r}_i - \vec{r}_j) = 0 \tag{35}$$

the derivative is

$$\frac{\partial F_{ijk}}{\partial X} = \left(\frac{\partial m_k}{\partial X} m_i + \frac{\partial m_i}{\partial X} m_k\right) c_{ki} - \left(\frac{\partial m_k}{\partial X} m_j + \frac{\partial m_j}{\partial X} m_k\right) c_{jk} + \\ + \left(\frac{\partial m_j}{\partial X} m_i + \frac{\partial m_i}{\partial X} m_j\right) c_{ij} - 2 m_i \frac{\partial m_i}{\partial X} \tag{36}$$

that can be written as

$$\frac{\partial F_{ijk}}{\partial X} = \frac{\partial m_i}{\partial X}(m_k\, c_{ki} + m_j\, c_{ij} - 2m_i) + \\ + \frac{\partial m_j}{\partial X}(m_i\, c_{ij} - m_k\, c_{jk}) + \frac{\partial m_k}{\partial X}(m_i\, c_{ki} - m_j\, c_{jk}) \tag{37}$$

Substituting for the derivatives the expression given in Eq. (28) we obtain

$$\frac{\partial F_{ijk}}{\partial X} = (X - X_i)\left(\frac{m_k}{m_i}\, c_{ki} + \frac{m_j}{m_i}\, c_{ij} - 2\right) + \\ + (X - X_j)\left(\frac{m_i}{m_j}\, c_{ij} - \frac{m_k}{m_j}\, c_{jk}\right) + \\ + (X - X_k)\left(\frac{m_i}{m_k}\, c_{ki} - \frac{m_j}{m_k}\, c_{jk}\right) \tag{38}$$

The derivatives $\frac{\partial F_{ijk}}{\partial Y}$ and $\frac{\partial F_{ijk}}{\partial Z}$ can be easily derived from Eq. (38) by substituting $[X, X_i, X_j, X_k]$ with $[Y, Y_i, Y_j, Y_k]$ and $[Z, Z_i, Z_j, Z_k]$, respectively. Analogously, we have

$$\frac{\partial F_{jki}}{\partial X} = (X - X_j)\left(\frac{m_i}{m_j}\, c_{ij} + \frac{m_k}{m_j}\, c_{jk} - 2\right) + \\ + (X - X_k)\left(\frac{m_j}{m_k}\, c_{jk} - \frac{m_i}{m_k}\, c_{ki}\right) + \\ + (X - X_i)\left(\frac{m_j}{m_i}\, c_{ij} - \frac{m_k}{m_i}\, c_{ki}\right) \tag{39}$$

and

$$\frac{\partial F_{kij}}{\partial X} = (X - X_k)\left(\frac{m_j}{m_k}\, c_{jk} + \frac{m_i}{m_k}\, c_{ki} - 2\right) + \\ + (X - X_i)\left(\frac{m_k}{m_i}\, c_{ik} - \frac{m_j}{m_i}\, c_{ij}\right) + \\ + (X - X_j)\left(\frac{m_k}{m_j}\, c_{jk} - \frac{m_i}{m_j}\, c_{ij}\right) \tag{40}$$

Again, the solving system is built as Eq. (32)

$$J\, \delta\vec{r} = \begin{bmatrix} \frac{\partial F_{123}}{\partial X} & \frac{\partial F_{123}}{\partial Y} & \frac{\partial F_{123}}{\partial Z} \\ \frac{\partial F_{231}}{\partial X} & \frac{\partial F_{231}}{\partial Y} & \frac{\partial F_{231}}{\partial Z} \\ \frac{\partial F_{312}}{\partial X} & \frac{\partial F_{312}}{\partial Y} & \frac{\partial F_{312}}{\partial Z} \\ \vdots & \vdots & \vdots \end{bmatrix} \left\{ \begin{array}{c} \delta X \\ \delta Y \\ \delta Z \end{array} \right\} = - \left\{ \begin{array}{c} F_{123} \\ F_{231} \\ F_{312} \\ \vdots \end{array} \right\} = -\vec{F} \tag{41}$$

and Eq. (33) is then used to obtain the least square solution. The method solving Eq. (21) usually requires much more iterations than the method solving Eq. (22). For this convergence speed reason, in the numerical tests section, the first approach is used.

Alternatively, it is possible to build a numerical technique that compute all the real roots of Eq. (21) or, equivalently, of Eq. (22). This second procedure, which can be used in the *lost-in-space* case (no attitude and position estimations available) as well as in the recursive case (approximate attitude and position estimations available), uses the *resultant* to eliminate variables from the attitude-free systems of nonlinear polynomial equations given in Eq. (21). This hybrid numerical/symbolic technique to compute all the real roots of Eq. (21) or, equivalently, of Eq. (22), is described in the following section.

Resultants, Contour Integrals, and 3×3 Quadratic Systems

From the last section, it is clear that if we can solve the system of equations

$$\begin{cases} \alpha^2 + \beta^2 - 2\alpha\beta\, c_{ij} - d_{ij}^2 = 0 \\ \beta^2 + \gamma^2 - 2\beta\gamma\, c_{jk} - d_{jk}^2 = 0 \\ \gamma^2 + \alpha^2 - 2\gamma\alpha\, c_{ki} - d_{ki}^2 = 0 \end{cases}$$

for α, β, γ, then we obtain an estimation of the camera position. In order to solve the above system of equations, we will use the *toric resultant* [16, 18], which is an efficient and modern generalization of the classical *Sylvester Resultant* of the 19th century. So let us first briefly recall the older resultant before introducing the modern version we use.

The classical *Sylvester Resultant* gives us a convenient way to eliminate one variable from any pair of univariate polynomial equations. For instance, given polynomials $f_g(\alpha) := g_0 + g_1\alpha + \cdots + g_m\alpha^m$ and $f_h(\alpha) := h_0 + h_1\alpha + \cdots + h_\ell\alpha^\ell$, it is known [15] that the $(m+\ell) \times (m+\ell)$ determinant

$$\mathrm{Det}\begin{bmatrix} h_0 & \cdots & h_{\ell-1} & h_\ell & 0 & 0 & \cdots & 0 \\ 0 & h_0 & \cdots & h_{\ell-1} & h_\ell & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & h_0 & \cdots & h_{\ell-1} & h_\ell \\ g_0 & \cdots & g_{m-1} & g_m & 0 & 0 & \cdots & 0 \\ 0 & g_0 & \cdots & g_{m-1} & g_m & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & g_0 & \cdots & g_{m-1} & g_m \end{bmatrix}$$

(with exactly m "g" rows and exactly ℓ "h" rows) vanishes iff f_g and f_h have a common complex root *or* their leading coefficients (g_m and h_ℓ) both vanish. The above determinant is known as the Sylvester Resultant and, with a little extra work, yields a method to eliminate a single variable from any pair of bivariate equations. In greater generality (see, e.g., [16]), there is a resultant associated to any system of $n+1$ polynomial equations in n variables with parameteric equations. By treating roots at infinity correctly (see, e.g., [17]), one can use these higher-dimensional resultants to solve any system of n polynomial equations in n variables. However, higher-dimensional resultants are still rather subtle [16].

Fortunately, our system at hand is 3×3, and Amit Khetan has recently discovered an elegant determinental formula for certain 3×2 resultants with many algorithmic applications [18]. For example, applying his formula to our system above yields that $(\alpha, \beta, \gamma) \in \mathbb{C}^3$ is a root iff

$$P(\gamma) := \det \begin{bmatrix} \rho_{123} & -\rho_{134} & \rho_{126}-\rho_{135} & 0 & \rho_{146} & \rho_{156} \\ \rho_{125} & \rho_{145} & \rho_{146}-\rho_{235} & 0 & \rho_{246}+\rho_{345} & \rho_{346}+\rho_{256} \\ \rho_{126} & \rho_{146} & \rho_{156}-\rho_{236} & 0 & \rho_{346} & \rho_{356} \\ -d_{ij}^2 & 0 & 0 & 1 & -2c_{ij} & 1 \\ \gamma^2-d_{jk}^2 & 0 & -2\gamma c_{jk} & 0 & 0 & 1 \\ \gamma^2-d_{ki}^2 & -2\gamma c_{ki} & 0 & 1 & 0 & 0 \end{bmatrix} = 0 \quad (42)$$

where ρ_{ijk} indicates the determinant of the 3×3 matrix formed by taking the coefficient vectors formed by the $i^{\underline{\text{th}}}$, $j^{\underline{\text{th}}}$, and $k^{\underline{\text{th}}}$ columns of the bottom three rows of our matrix above. E.g., ρ_{135} is

$$\rho_{135} = \det \begin{bmatrix} -d_{ij}^2 & 0 & -2c_{ij} \\ \gamma^2-d_{jk}^2 & -2\gamma c_{jk} & 0 \\ \gamma^2-d_{ki}^2 & 0 & 0 \end{bmatrix}$$

In particular, we obtain a single (univariate) polynomial encoding the γ coordinates of all our solutions. Such a polynomial is usually called an *eliminant*, and is usually the result of a resultant calculation where the underlying coefficients depend on a single parameter.

Example: Taking a randomly generated example, the parameters in our system above specialize as follows

$$d_{ij}:=74.4566,\ d_{jk}:=26.7947,\ d_{ki}:=43.9924,\ c_{ij}:=0.3089,\ c_{jk}:=0.5807,\ c_{ki}:=0.8581$$

Equation (42) for our *eliminant* becomes

$$P(\gamma) = \det \begin{bmatrix} \rho_{123} & -\rho_{134} & \rho_{126}-\rho_{135} & 0 & \rho_{146} & \rho_{156} \\ \rho_{125} & \rho_{145} & \rho_{146}-\rho_{235} & 0 & \rho_{246}+\rho_{345} & \rho_{346}+\rho_{256} \\ \rho_{126} & \rho_{146} & \rho_{156}-\rho_{236} & 0 & \rho_{346} & \rho_{356} \\ -5543.8 & 0 & 0 & 1 & -0.61779 & 1 \\ \gamma^2-717.96 & 0 & -1.1615\gamma & 0 & 0 & 1 \\ \gamma^2-1935.3 & -1.7162\gamma & 0 & 1 & 0 & 0 \end{bmatrix} = 0$$

Note in particular that some of the ρ_{ijk} terms involve the variable γ. With a little algebra, it is not difficult to see that $P(\gamma)$ will always be of degree 8 in γ. We also point out that while a full symbolic expansion of the above determinant is unwieldy, the above determinant is *very* easy to evaluate at any given numerical γ. Furthermore, the algorithm we are about to describe to solve $P(\gamma)=0$ uses *only* evaluations of P (none of its derivatives), thus taking advantage of the low evaluative complexity of P.

From here, we then clearly need a reliable method to find the real roots of a degree 8 polynomial. The approach we derive here is the following: use a variant of Cauchy's residue theorem [12] to first coarsely isolate the real roots, then apply Newton method to get any further precision.

In particular, the following formula is particularly efficient: First, let $R \subset \mathbb{C}$ be any "short and fat" rectangle in the complex plane containing an interval I in the real numbers $\mathbb{R}$, and let $u : [0,1] \longrightarrow \partial R$ be any parametrization of the boundary of R. Then, for N sufficiently large,[**] the nearest integer to

$$\frac{1}{4\pi}\sum_{i=1}^{N} \mathrm{ArcSin}\left(\mathrm{Im}\left(\frac{\bar{P}\left(u\left(\frac{i-1}{N}\right)\right) P\left(u\left(\frac{i}{N}\right)\right)}{P\left(u\left(\frac{i-1}{N}\right)\right) \bar{P}\left(u\left(\frac{i}{N}\right)\right)}\right)\right)$$

where Im and $\bar{(\cdot)}$ respectively denote imaginary part and complex conjugate, is exactly the number of complex roots of P in the interior of the rectangle R.[††]

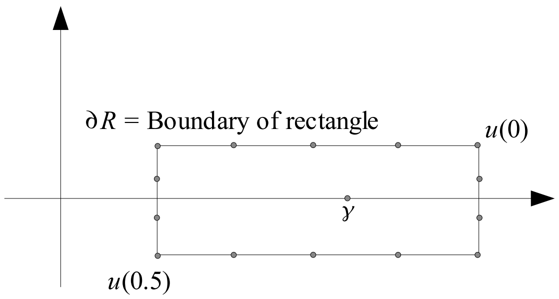

Figure 2: A discretization of the Cauchy Integral Theorem.

In particular, if R is "short" enough, the number of roots in R is exactly the number of roots in the interval I. One can then use this formula to quickly find small intervals containing all the real roots of P, via the ancient algorithm of bisection. In particular, the reason we use the formula above instead of the usual Sturm-Habicht sequence techniques is that the latter techniques require the coefficients of P, which are not readily available and will most likely suffer from round-off error anyway.

[**] In the "lost-in-space" setting, N is usually no more than a few hundred.

[††] This property follows immediately from the Cauchy Residue Theorem [12], after a routine integration.

Narrowing the intervals to some pre-determined tolerance, we can then switch to Newton's method to quickly get even better approximations to all the real roots. From here, we then apply the quadratic formula to two of our original equations

$$\begin{cases} \beta^2+\gamma^2-2\beta\gamma\, c_{jk}-d_{jk}^2=0 \\ \gamma^2+\alpha^2-2\gamma\alpha\, c_{ki}-d_{ki}^2=0 \end{cases}$$

to solve for α and β. In particular, a quick sign evaluation of the discriminant immediately tells us which roots of our system in $\mathbb{C}^3$ in fact lie in $\mathbb{R}^3$. Finally, once the camera position has been estimated, we can then estimate the camera attitude by using ESOQ-2 [13, 14], the fastest existing attitude estimation algorithm fully complying with the Wahba's definition of optimal attitude[9].

In closing, we point out that, much like was done in [19] for sparse polynomial systems in arbitrarily many variables, one can actually derive completely rigorous high probability estimates of the complexity of our algorithm. The latter estimates will appear in a forthcoming paper.

Numerical Tests

A Monte Carlo approach has been used to test the loss of accuracy of the LARA and the FAFA proposed methods. The numerical tests performed are not exhaustive, and they should be intended as a first step to prove the feasibility of these two methods. Two sets of $1,000$ numerical tests have been performed characterized by two different measurement noise levels associated with very accurate measurements ($\sigma = 0.001°$) and less accurate ($\sigma = 0.05°$). The camera position has been chosen to be randomly located 100 meters far from the origin of coordinates. The position of the "*center of beacons*" has been chosen to be randomly located between $d = 50m$ and $d = 100m$ far from the camera position. The field of view of the camera has been selected to be $\vartheta_{FOV} = 20°$ of aperture, even though VISNAV can even reach 75°. The positions of the $N > 4$ beacons are randomly chosen to be within a ball centered at the "*center of beacons*" whose radius is $r_{ball} = d \sin\vartheta_{FOV}$.

Figure 3 shows the attitude and the position errors obtained in $1,000$ random tests performed with a measurement noise of $\sigma = 0.001°$, while the results given in Fig. 4 are associated with a measurement noise of $\sigma = 0.05°$. The position errors are represented by the distances between the true and the estimated positions, while the attitude errors are represented by the maximum angular direction error

$$\varepsilon = \cos^{-1}\left(\frac{\mathrm{tr}\Delta - 1}{2}\right)$$

This error is the angular error associated with a direction orthogonal to the principal axis of the corrective attitude matrix

$$\Delta = T\, C^{\mathrm{T}}$$

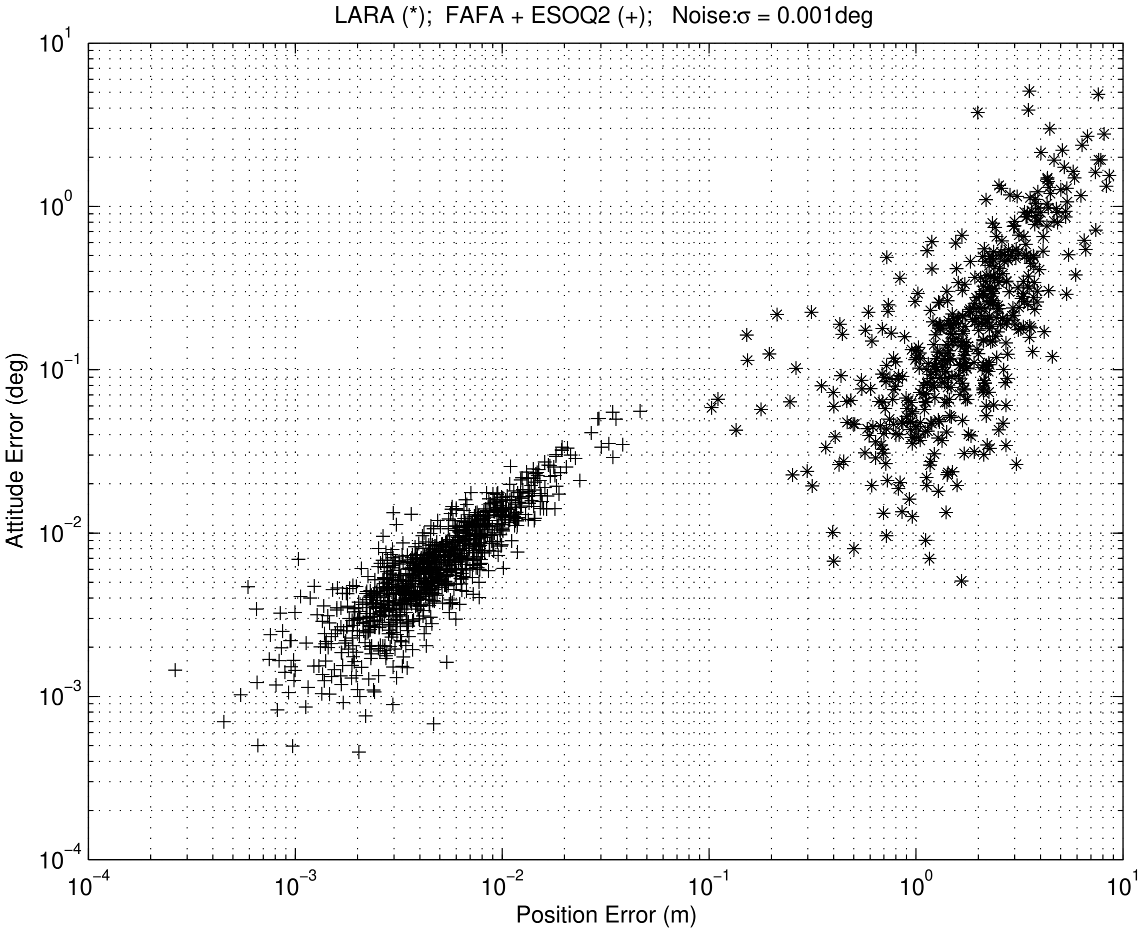

Figure 3: Numerical tests results with $\sigma = 0.001°$

where T and C represent the true and the estimated attitudes, respectively.

Figures 3 and 4 clearly shows that *First Attitude Free Approach* is better behaved than *Linear Algebra Resection Approach*, for the latter method gives unacceptably large errors in the case of measurement noise around $\sigma = 0.05°$. However, the LARA method deserves particular attention for two main reasons. First because it can be used in the Lost-In-Space case, and second because it is a method that is subject of many improvements, which will be the subject of a future paper.

As for SAFA, we note that SAFA is specifically designed for the case of 3 beacons and preliminary numerical experiments show that it is quite efficient. In a future version of this paper, we will extend SAFA to an arbitrary number of beacons, and present extensive numerical results.

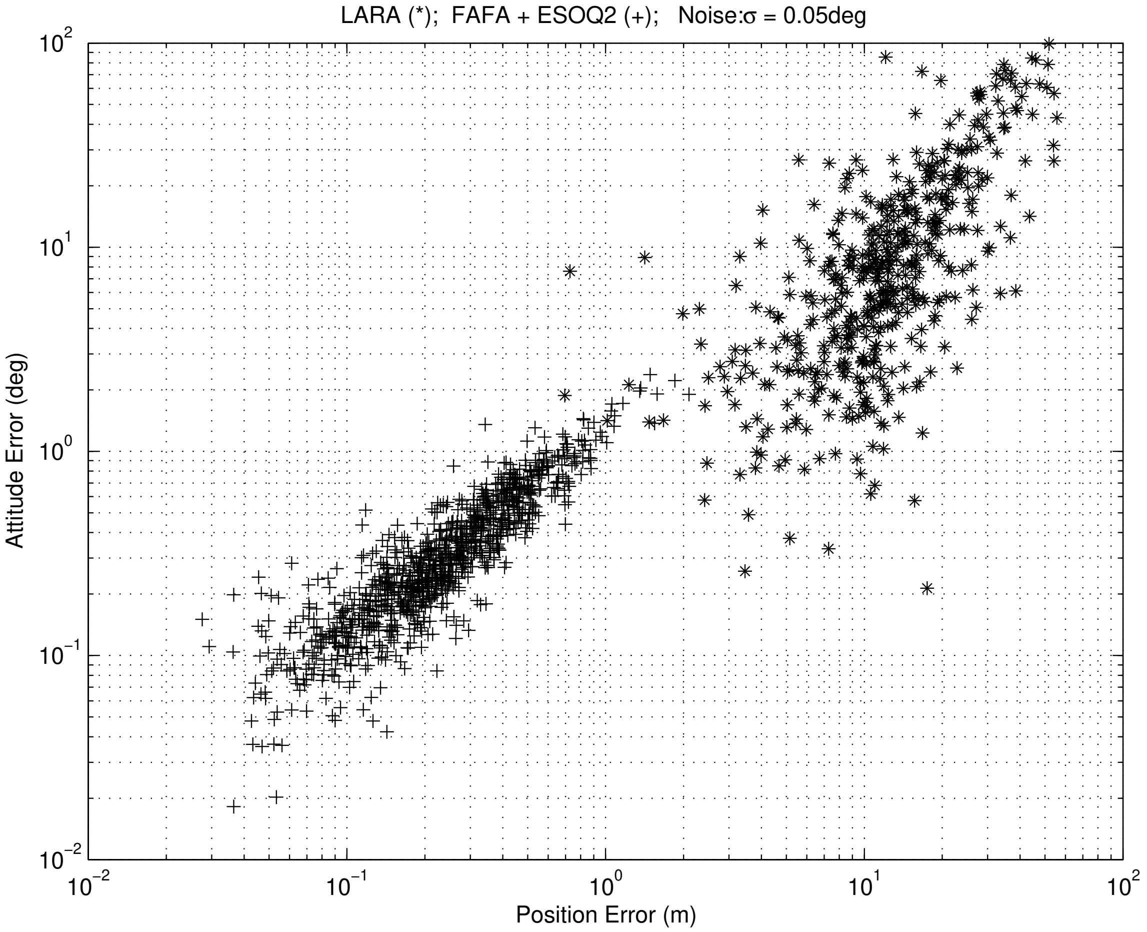

Figure 4: Numerical tests results with $\sigma = 0.05°$

Conclusion

This paper introduces three novel approaches to solve the attitude and position estimation problem based on the observation of a set of N directions, using a vision-based technology camera. Two of these methods can be applied in the *lost-in-space* case, where no approximate solution is available, while the third method is particularly suitable when attitude and position are constantly and frequently updated.

The first approach, called *Linear Algebra Resection Approach* (LARA), finds the solution as the left eigenvector associated with the minimum singular value of a rectangular data matrix.

The second and third approaches (FAFA and SAFA) recast the problem into a nonlinear system of equations in terms of the unknown position only, and then solve for the attitude using an existing optimal attitude estimator.

FAFA uses a least-square Newton-Raphson iterative procedure while SAFA uses the *toric resultant* to reduce the problem to a single degree 8 polynomial in one variable, and then employs a Newton's method and a discretization of the Cauchy integral theorem to quickly isolate just the real solutions.

Robustness of the proposed methods with respect to measurement noise is justified through numerical tests.

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