Feasibility of Circuit Polynomials without Black Swans

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ABSTRACT

Suppose f is a polynomial in n variables with degree d, exactly $n+k$ monomial terms, coefficients in $\{\pm 1, \ldots, \pm H\}$ for some $H \in \mathbb{N}$, and Newton polytope of positive volume. Testing real feasibility of such an f is a fundamental task whose bit-complexity remains a mystery, even in the first non-trivial case $k=2$: The fastest algorithms so far have deterministic bit-complexity $(n \log(dH))^{O(n)}$. We prove a significant speed-up that holds for all but a small collection of inputs in the $k=2$ case: Bit complexity $(n \log(dH))^{O(1)}$ for all but a $O\left(\frac{1}{2^n H}\right)$-fraction of the f above, for any fixed support. Our result follows by combining a connection to diophantine approximation with a more recent anti-concentration result. In particular, we show that for random inputs, Baker's famous theorem on linear forms in logarithms can be significantly sharpened. We also consider extensions beyond feasibility such as counting connected components.

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1 INTRODUCTION AND MAIN RESULTS

Counting the number of connected components (a.k.a. *pieces*) for the positive zero set, $Z_+(f)$, of a Laurent polynomial f, as a function of its monomial term structure, is a fundamental problem from real algebraic geometry that is still far from completely understood. This is unfortunate, because many real zero sets occuring in practice come from highly structured polynomials, and one of the most basic structures to consider is monomial term structure.

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Definition 1.1. *Suppose $f \in \mathbb{Z}\left[x_1^{\pm 1}, \ldots, x_n^{\pm 1}\right]$ is of the form $f(x) = \sum_{i=1}^{n+k} c_i x^{a_i}$, where $c_i \in \{\pm 1, \ldots, \pm H\}$, $a_i \in \{-d, \ldots, d\}^{n\times 1}$, and $x^{a_i} := x_1^{a_{1,i}} \cdots x_n^{a_{n,i}}$ for all i. Assuming in addition that $A := \{a_1, \ldots, a_{n+k}\}$ has cardinality $n+k$, we call such an f an* n-variate $(n+k)$-nomial *of type (A, d, H), and say that f is* supported *on A. If we also have that the matrix $\widehat{\mathcal{A}} := \begin{bmatrix} 1 & \cdots & 1 \\ a_1 & \cdots & a_{n+k} \end{bmatrix} \in \mathbb{Z}^{(n+1)\times(n+k)}$ has rank $n+1$ then we call f an* honest *n-variate $(n+k)$-nomial.* ⋄

Remark 1.2. *The geometric restriction on the exponent vectors (via the rank of $\widehat{\mathcal{A}}$ above) makes the parameter n meaningful: Without this restriction, one could find a simple change of variables to reduce to a smaller n while still preserving $n+k$, e.g., the positive roots of $1 - xy + x^{100}y^{100}$ can be determined from the positive roots of $1 - u + u^{100}$ by substituting $u = xy$. Note also that the rank restriction on $\widehat{\mathcal{A}}$ forces $k \geq 1$.* ⋄

For the special case $n=1$, Descartes' Rule tells us that the number of pieces (for the positive zero set of a univariate $(k+1)$-nomial) is at most k, and this bound is tight thanks to the explicit family of examples $(x_1 - 1)(x_1 - 2)\cdots(x_1 - k)$. However, for $n=2$, it isn't even known if the number of pieces admits an upper bound of the form $k^{O(1)}$: The best upper bound is still exponential in k^2 [BS09], and no family of examples evincing even $\Omega(k^2)$ pieces is known. However, recent *probabilistic* results in real fewnomial theory [Kho91, BET-C19] suggest that, *on average* (for many natural coefficient distributions), the number of pieces should be $O(k^2)$.

Remark 1.3. *All O-, Ω-, and o-constants in our proofs are effective and absolute, i.e., they are actual constants that can be made explicit, albeit with some effort. Also, for an f as in Definition 1.1, we define the* size *of f to be*

$$\sum_{i=1}^{n+k}\left(\left\lceil \log_2(2+|c_i|)\right\rceil + \sum_{j=1}^{n}\left\lceil \log_2(2+|a_{i,j}|)\right\rceil\right),$$

i.e., the sum of the bit sizes of the coefficients and the exponents of f. So in our setting, polynomial-time algorithms have bit complexity $((n+k)\log(dH))^{O(1)}$, as opposed to many basic algorithms in classical computational algebra that have complexity $(dn\log H)^{O(n)}$. ⋄

For the *algorithmic* question of actually counting the number of pieces for a given $Z_+(f)$, our knowledge is even sparser: A polynomial-time algorithm is known only for the cases $(n,k) \in \{(1,1),(1,2)\}$ [BRS09, Bih11]. There is also the folkloric fact that the case $k=1$ and n arbitrary never results in $Z_+(f)$ having more than 1 piece, and (for $k=1$) deciding between 0 and 1 pieces is doable in time $O(n)$ (see, e.g., Lemma 2.14 in Section 2.1 below). On the other hand, we have **NP**-hardness (for deciding non-emptiness) if we fix any $\varepsilon > 0$, let $n \longrightarrow \infty$, and take $k = n^{\varepsilon}$ [BRS09]. What happens between $k=1$ and $k=n^{\varepsilon}$ is still a mystery.

In particular, the fastest algorithms for just deciding if there are any pieces at all (for $k=2$ and n arbitrary) have deterministic bit-complexity $(n\log(dH))^{O(n)}$ [BRS09]. (See also [BPR06, BR14] for much more powerful and general algorithms which, unfortunately, are no faster in the $k=2$ case.) So we prove the following significant speed-up for the case $k=2$:

Theorem 1.4. *Following the notation above, for a fraction of $1-O\left(\frac{1}{2^n H}\right)$ of honest n-variate $(n+2)$-nomials of type (A,d,H), we can decide whether $Z_+(f)$ is empty in deterministic time $O(n^{3.373}\log^3(ndH))$.*

Theorem 1.4 is proved in Section 3. The first step is to reduce our counting question to a Diophantine problem: Determining the sign of an integer linear combination of logarithms of integers. This reduction is straightforward, after we review a special case of the $\mathcal{A}$-discriminant [GKZ94] in Section 2. So the main subtlety is understanding the peculiar behavior of linear forms in logarithms.

By combining with a more refined classification of isotopy types from [BDPRRR24], we can in fact *count* the number of pieces of $Z_+(f)$ within the same time bound as Theorem 1.4, and this addendum is currently being finalized. However, the underlying probabilistic technique is the same for both results, so we cover it now.

1.1 Probabilistic Bounds on Linear Forms in Logarithms

A landmark 1966 result in transcendental number theory due to Baker can be coarsely summarized as follows [Bak77]: Let $\Lambda(b,c) := \sum_{i=1}^m b_i \log c_i$ be an integer linear combination of logarithms of rational numbers. Let H denote the maximal absolute value among the integers that appear in the numerators and denominators of the c_i, and let $B := \max_i |b_i|$. A special case of *Baker's Theorem on Linear Forms in Logarithms* [Bak77, BW93, Mat00, Nes03] then implies

$$\Lambda(b,c) \neq 0 \Rightarrow \log|\Lambda(b,c)| > -O(\log H)^m \log B. \qquad (1)$$

This bound proved remarkably difficult to prove, and the special case $m=2$ already implies the solution to Hilbert's Seventh Problem (proving that a^b is transcendental for $a \notin \{0,1\}$ algebraic and b algebraic and irrational).

Baker won a Fields Medal in 1970 for his lower bound, and later his bound also proved useful for computing explicit upper bounds for the size of integer points on curves of genus 1 (see, e.g., [Sch92]). More recently, Baker's lower bound has also found use in the design and analysis of algorithms for real algebraic geometry and parsing (see, e.g., [BRS09, BHPR11, BSY14, Roj22]).

It has been conjectured that Baker's lower bound is far from sharp: Lang and Waldschmidt used a simple heuristic argument to motivate a conjecture that the optimal bound should be $-O(m\log(HB))$ [Lan78, Pg. 213]. However, there appears to have been no progress whatsoever, for close to half a century, on their conjecture. Our algorithmic goals happen to naturally motivate a probabilistic approach to this bound: Can we prove a sharper version of Baker's lower bound, for *most* inputs instead, and thereby prove that earlier algorithms for real feasibility can be sped up most of the time?

Remark 1.5. *The title of our paper was inspired by the title of [AL17], which studies algorithms that are fast on average, outside of a small region of inputs. In contrast, we study a setting where* worst-case *complexity is low outside of a small region of inputs.* ⋄

1.2 Two Models of Discrete Randomness

We start by stating an important consequence of Corollary 1.4 from an elegant paper of Rudelson and Vershynin [RV15].

Lemma 1.6. *Consider a random vector $X = (X_1, \ldots, X_m)$ where the X_i are independent random variables. Let $p, t > 0$ be parameters such that $\sup_{z\in\mathbb{R}} \mathbb{P}\{|X_i - z| \leq t\} \leq p$ for all $i \in \{1,\ldots,m\}$. Then there is a constant $C>0$ such that for any fixed $b := (b_1,\ldots,b_m)$ we have*

$$\sup_{z\in\mathbb{R}} \mathbb{P}\{|\langle X, b\rangle - z| \leq t\,\|b\|_2\} \leq Cp. \qquad \blacksquare$$

Remark 1.7. *To obtain the lemma one uses the special case $d=1$ of [RV15, Cor. 1.4] and notes that the corollary is stated for a projection where we allow taking inner product with an arbitrary vector b.* ⋄

1.2.1 Random Integers with Controlled Bit-Size.

Proposition 1.8. *Let $H \in \mathbb{N}$ and consider a uniformly random α chosen from $\{\pm1,\ldots,\pm H\}$. Then, for any $z \in \mathbb{R}$ and any $\varepsilon \in \left(\frac{1}{H}, \frac{1}{4}\right)$ we have:*

$$\mathbb{P}\{|\log|\alpha| - z| \leq \varepsilon\} \leq 9\varepsilon.$$

Proof: Note that for any interval $[s,t]$ we have that

$\mathbb{P}\{\log|\alpha| \in [s,t]\} = \mathbb{P}\{|\alpha| \in [e^s, e^t]\}$, which is in turn bounded from above by $\frac{\text{Twice the number of integers in } [e^s,e^t]\cap[1,\ldots,H]}{2H}$.

So if $e^{z-\varepsilon} > H$ then we have $\mathbb{P}\{|\log \alpha - z| \leq \varepsilon\} = 0$. If $e^{z-\varepsilon} \leq H$ then we have

$$\mathbb{P}\{|\log\alpha - z| \leq \varepsilon\} \leq \frac{1+\left(e^{z+\varepsilon} - e^{z-\varepsilon}\right)}{2H} \leq \frac{2}{2H} + 2\left(e^{2\varepsilon}-1\right)\frac{e^{z-\varepsilon}}{2H}.$$

For any $\varepsilon < \frac{1}{4}$ Taylor's expansion yields $e^{2\varepsilon} - 1 \leq \frac{1}{1-2\varepsilon} - 1 = \frac{2\varepsilon}{1-2\varepsilon} \leq 4\varepsilon$. So we have

$$\mathbb{P}\{|\log\alpha - z| \leq \varepsilon\} \leq \frac{2}{2H} + 8\varepsilon\left(\frac{e^{z-\varepsilon}}{2H}\right) \leq 9\varepsilon. \qquad \blacksquare$$

Remark 1.9. *The probability distribution in Proposition 1.8 was chosen for simplicity. Similar estimates can be easily derived for more general distributions, e.g., the uniform distribution on $\{x-H,\ldots,-1,1,x,\ldots,x+H\}$ for any $0 \leq x \leq H$.* ⋄

Combining Lemma 1.6 and Proposition 1.8 gives the following probabilistic estimate on integer linear sums of logs.

Corollary 1.10. *Fix any $b := (b_1,\ldots,b_m) \in (\mathbb{Z}\setminus\{0\})^m$ and let $\alpha = (\alpha_1,\ldots,\alpha_m)$ be a uniformly random vector in $\{\pm1,\ldots,\pm H\}^m$. Then there is a constant $C_0 > 0$ such that for any $z \in \mathbb{R}$ and any $\varepsilon \in \left(\frac{1}{H}, \frac{1}{4}\right)$ we have:*

$$\mathbb{P}\{|b_1\log|\alpha_1| + \cdots + b_m\log|\alpha_m| - z| \leq \varepsilon|b|_2\} \leq C_0\varepsilon$$

Since $b_i^2 \geq 1$ for all i, this also yields

$$\mathbb{P}\{|b_1\log|\alpha_1| + \cdots + b_m\log|\alpha_m| - z| \leq \varepsilon\sqrt{m}\} \leq C_0\varepsilon. \qquad \blacksquare$$

Remark 1.11. *As noted in Remark 1.9, we could also mimic the approach above for independent a_i uniformly distributed in $\{x_i - H, \ldots, -1, 1, \ldots, x_i + H\}$, for arbitrary integers x_i satisfying $0 \leq x_i \leq H$. This can be thought of as an adversarial random model where the adversary is allowed to pick a bad $x = (x_1, \ldots, x_n)$ and a random perturbation with magnitude $H \geq \max_i |x_i|$ is added to x to obtain the random vector a.* $\diamond$

Remark 1.12. *The constant C_0 in Corollary 1.10 satisfies $C_0 \leq 9C$ where C is the constant from Lemma 1.6. The smallest suitable C known so far is $\sqrt{2}$, thanks to [LPP16].* $\diamond$

1.2.2 Discrete Gaussians. A distribution that is commonly used in discrepancy theory and integer programming applications is the discrete Gaussian (see, e.g., [ADRSD15]).

Here we will consider discrete Gaussian centered at an arbitrary integer $a \in \mathbb{Z}$ with standard deviation H. More precisely, for any $x \in \mathbb{Z}$ we define $p(x) := e^{-\frac{(x-a)^2}{2H^2}}$ and set $Q := \sum_{x\in\mathbb{Z}} e^{-\frac{(x-a)^2}{2H^2}}$. Then the discrete Gaussian centered at a with standard deviation H is the random variable X that takes integer values with the following weights: $\mathbb{P}(X = x) := \frac{p(x)}{Q}$.

First we make a quick computation:

$$1 + Q = 2\sum_{y=0}^{\infty} e^{-\frac{y^2}{2H^2}} \geq 2\int_0^{\infty} e^{-\frac{y^2}{2H^2}}\, dy = \int_{-\infty}^{\infty} e^{-\frac{y^2}{2H^2}}\, dy = H\sqrt{2\pi}$$

So $Q > H\sqrt{2\pi} - 1$, and for any $H > 4$, this also yields $Q > 2H$.

Lemma 1.13. *Let a be an arbitrary integer and let X be the discrete Gaussian centered at a with standard deviation H where $H > |a|$. Then, for any real number z and any $\frac{1}{4} > \varepsilon > \frac{1}{H}$ we have*

$$\mathbb{P}\{|\log|X| - z| \leq \varepsilon\} \leq 9\varepsilon.$$

Proof: Note that for any interval $[s, t]$ the probability we have

$$\mathbb{P}\{\log|X| \in [s,t]\} = \mathbb{P}\{|X| \in [e^s, e^t]\} \leq \frac{2\sum_{y\in[e^s,e^t]\cap\mathbb{Z}} e^{-\frac{(y-a)^2}{2H^2}}}{Q}$$

So we have

$$\mathbb{P}\{|\log X - z| \leq \varepsilon\} \leq \frac{\sum_{y\in[e^{z-\varepsilon},e^{z+\varepsilon}]\cap\mathbb{Z}} e^{-\frac{(y-a)^2}{2H^2}}}{H}$$

For any $\varepsilon < \frac{1}{4}$ Taylor's expansion yields $e^{2\varepsilon} - 1 \leq \frac{1}{1-2\varepsilon} - 1 = \frac{2\varepsilon}{1-2\varepsilon} \leq 4\varepsilon$. So, we have $e^{z+\varepsilon} - e^{z-\varepsilon} \leq 4\varepsilon\, e^{z-\varepsilon}$. If $H \geq a \geq e^{z-\varepsilon}$ we have

$$\mathbb{P}\{|\log X - z| \leq \varepsilon\} \leq \frac{4\varepsilon e^{z-\varepsilon} + 1}{H} \leq 9\varepsilon$$

If $a < e^{z-\varepsilon}$, let $d =: \min_{y\in[e^{z-\varepsilon},e^{z+\varepsilon}]\cap\mathbb{Z}} |y - a|$. Note that $a + d > e^{z-\varepsilon}$, and thus we have $e^{z+\varepsilon} - e^{z-\varepsilon} \leq 4\varepsilon(a + d)$. So, we have

$$\mathbb{P}\{|\log X - z| \leq \varepsilon\} \leq \frac{(4\varepsilon(a+d)+1)e^{-\frac{d^2}{2H^2}}}{H} \leq \varepsilon + 4\varepsilon\frac{(a+d)e^{-\frac{d^2}{2H^2}}}{H}$$

If $d < H$, then we are done. Let $\delta = \frac{d}{H} > 1$, then we have

$$\mathbb{P}\{|\log X - z| \leq \varepsilon\} \leq \left(1 + 8\delta e^{-\frac{\delta^2}{2}}\right)\varepsilon \leq 9\varepsilon \qquad \blacksquare$$

Combining Lemmata 1.6 and 1.13 immediately gives the following probabilistic estimate on integer linear sums of logs.

Corollary 1.14. *Let $b = (b_1, \ldots, b_m) \in \mathbb{Z}^m$ be an integer vector with $b_i \neq 0$. Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a random vector where the α_i are independent discrete Gaussian random variables centered at integers x_i with variances H_i where $H_i > |x_i|$. Let $H := \max_{1\leq i\leq m} H_i$, then for any $z \in \mathbb{R}$ and $\varepsilon \in \left(\frac{1}{H}, \frac{1}{4}\right)$ we have*

$$\mathbb{P}\{|b_1 \log|\alpha_1| + \cdots + b_m \log|\alpha_m| - z| \leq \varepsilon|b|_2\} \leq C_0\varepsilon.$$

(C_0 being the constant from Corollary 1.10.) Since $b_i^2 \geq 1$ for all i, this also yields

$$\mathbb{P}\{|b_1 \log|\alpha_1| + \cdots + b_m \log|\alpha_m| - z| \leq \varepsilon\sqrt{m}\} \leq C_0\varepsilon. \ \blacksquare$$

There is evidence [Roj22] that our approach to new probabilistic speed-ups can be extended to *systems* of circuit polynomials (all with the same support). Technically, counting pieces in our setting here is accomplished by using signs of linear combinations of logarithms of rational numbers to decide which discriminant chamber contains f. (This is explained further in Section 2.) To count real solutions of circuit systems instead, there is a reduction (using *Gale Dual form* [BS07]) to counting real roots of a linear combination of logarithms of *degree one polynomials.* While the latter problem appears to be purely transcendental, one can reduce it (as analyzed in [Roj22]) to computing several signs of linear forms of logarithms of *real algebraic numbers.* So we can extend our approach to systems provided we have sufficiently strong extensions of Corollaries 1.10 and 1.14 to real algebraic numbers. (There are other technical hurdles as well, but we leave the details for future work.) This motivates Corollary 1.18 below as a first step toward the harder problem of counting real roots of circuit systems.

1.3 Random Algebraic Integers

Suppose we are given a primitive element u with a degree d field extension $\mathbb{Q}(u)$. We would like to generate random algebraic integers from the number field $\mathbb{Q}(u)$ that have height at most H. We first recall the basics.

Definition 1.15 (Height). *Let α be an algebraic number with minimal polynomial $c_0 + c_1 x + \cdots + c_d x^d$, and let $\alpha_1, \ldots, \alpha_d$ be the conjugates of α. Then the height of α, denoted by $H(\alpha)$ is defined as*

$$H(\alpha) := \left(|c_0| \prod_{i=1}^{d} \max\{1, \alpha_i\}\right)^{\frac{1}{d}}$$

The logarithmic height is defined as $h(\alpha) := \log H(\alpha)$.

The following lemma is standard (see, e.g., [BG06, Ch. 1]).

Lemma 1.16. *Let $x = c_0 + c_1 u + \cdots + c_{d-1} u^{d-1} \in \mathbb{Q}(u)$ where $\mathbb{Q}(u)$ is a degree d number field, then*

$$h(x) \leq d\left(\max_i h(\alpha_i) + h(u)\right) + \log d$$

Now we consider the following model of randomness: We generate $x = \xi_0 + \xi_1 u + \cdots + \xi_{d-1} u^{d-1}$ where ξ_i are independent discrete Gaussian random variables centered at arbitrary integers c_i and variances H_i where $H_i \geq |c_i|$. Let $H := \max\{H_0, \ldots, H_{d-1}, H(u)\}$, then by Markov's inequality and Lemma 1.16 we have

$$\mathbb{P}(h(x) \leq 4dH + 2\log d) \geq \frac{1}{2}$$

So we condition on x satisfying

$$h(x) \leq 4dH + 2\log d \tag{2}$$

and thus our randomness model is a uniform sample among the algebraic numbers $x = \xi_0 + \xi_1 u + \cdots + \xi_{d-1} u^{d-1}$ that satisfy (2). We call this model of randomness $\Gamma(c_0, \ldots, c_{d-1}, H_0, \ldots, H_{d-1}, u)$.

Lemma 1.17. *Let u be algebraic number with $|u| \geq 1$, and $Q(u)$ being degree d. Let X be a random variable distributed according to $\Gamma(c_0, \ldots, c_{d-1}, H_0, \ldots, H_{d-1}, u)$. Then, for any $z \in \mathbb{R}$ and any $\varepsilon \in \left(\frac{1}{H}, \frac{1}{4}\right)$ we have*

$$\mathbb{P}\{|\log|X| - z| \leq \varepsilon\} \leq 9\varepsilon.$$

Proof: By Lemma 1.6 and the fact that $\left|u^k\right| \geq 1$, X satisfies small-ball estimates at least as strong as discrete Gaussian random variables. The rest of the proof follows as in the case of discrete Gaussians. ■

Corollary 1.18. *Fix $b = (b_1, \ldots, b_m) \in (\mathbb{Z} \setminus \{0\})^m$ and let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a random vector where each α_i is an independent random variable distributed according to $\Gamma(a_0, \ldots, a_{d-1}, H_0, \ldots, H_{d-1}, u)$. Then for any $z \in \mathbb{R}$ and $\varepsilon \in \left(\frac{1}{H}, \frac{1}{4}\right)$ we have*

$$\mathbb{P}\{|b_1 \log|\alpha_1| + \cdots + b_m \log|\alpha_m| - z| \leq \varepsilon |b|_2\} \leq C_0 \varepsilon.$$

(C_0 being the constant from Corollary 1.10.) Since $b_i^2 \geq 1$ for all i this also yields

$$\mathbb{P}\{|b_1 \log|\alpha_1| + \cdots + b_m \log|\alpha_m| - z| \leq \varepsilon \sqrt{m}\} \leq C_0 \varepsilon.$$

2 WHICH SIDE ARE YOU ON?: CIRCUIT DISCRIMINANTS AND THEIR SIGNS

Let us first recall a rational function of absolute values that is related to a particular class of $\mathcal{A}$-discriminant polynomials.

Definition 2.1. *Suppose $A = \{a_1, \ldots, a_{m+2}\} \subset \mathbb{Z}^n$ is such that $\widehat{\mathcal{A}} := \begin{bmatrix} 1 & \cdots & 1 \\ a_1 & \cdots & a_{m+2} \end{bmatrix} \in \mathbb{Z}^{(n+1)\times(m+2)}$ has distinct columns and rank $m+1$ for some $m \leq n$. Let $b \in \mathbb{Z}^{(m+2)\times 1}$ be any generator of the right $\mathbb{Z}$-nullspace of $\widehat{\mathcal{A}}$. We then call A a* non-degenerate *circuit if and only if b has no zero coordinates (and a* degenerate *circuit otherwise). Also, for any non-degenerate circuit $A \subset \mathbb{R}^n$ of cardinality $m+2$, and any nonzero real $c_1, \ldots, c_{m+2}$, we define $\Xi_A(c_1, \ldots, c_{m+2}) := \left(\prod_{i=1}^{m+2} |c_i/b_i|^{b_i}\right) - 1$.*

In our setting, the c_i will always be the coefficients of a polynomial f supported on the circuit A. So we will often abuse notation by writing $\Xi_A(f)$ instead of $\Xi_A(c_1, \ldots, c_{m+2})$, assuming $f(x) = c_1 x^{a_1} + \cdots + c_{m+2} x^{a_{m+2}}$. When restricted to a suitable orthant in $\mathbb{R}^{m+2}$, our Ξ_A is a monomial multiple of the *$\mathcal{A}$-discriminant polynomial* $\Delta_{\mathcal{A}}$ from [GKZ94, Ch. 9]. From the development of [GKZ94, Ch. 9] (restricted to $\mathbb{R}$) we have the following summary of the key properties of Ξ_A that we'll need:

Theorem 2.2. *Suppose $A = \{a_1, \ldots, a_{m+2}\} \subset \mathbb{Z}^n$ is a non-degenerate circuit of cardinality $m+2$, $f \in \mathbb{R}\left[x_1^{\pm 1}, \ldots, x_n^{\pm 1}\right]$ is supported on A, and $f(x) = c_1 x^{a_1} + \cdots + c_{m+2} x^{a_{m+2}}$. Then $Z_+(f)$ has a singularity if and only if $\Xi_A(f) = 0$ and $\mathrm{sign}(b_1 c_1) = \cdots = \mathrm{sign}(b_{m+2} c_{m+2})$. In particular, when $m = n$, such a $Z_+(f)$ has at most 1 singular point.* ■

Example 2.3. *$A = \{0, 2, 7\} \subset \mathbb{Z}^1$ is a non-degenerate circuit, and we see that a suitable $b \in \mathbb{Z}^3$ is $b = (5, -7, 2)^\top$ (taking $(\cdot)^\top$ to mean transpose). Theorem 2.2 then tells us that $f(x) := c_1 + c_2 x^2 + c_3 x^7$ has a degenerate positive root if and only if [[$c_1, c_3 > 0 > c_2$ or $c_1, c_3 < 0 < c_2$] and $\left|\frac{c_1}{5}\right|^5 \left|\frac{c_2}{-7}\right|^{-7} \left|\frac{c_3}{2}\right|^2 = 1$]. Note that the last equality is equivalent to*

$$5\log|c_1| - 7\log|c_2| + 2\log|c_3| = 5\log(5) - 7\log(7) + 2\log(2).$$

Note also that $\Xi_A(5 - 7x^2 + 2x^7) = 0$ here, and the unique degenerate root of $5 - 7x^2 + 2x^7$ is 1. ⋄

Example 2.4. *$A = \{(0,0), (2,2), (7,7)\} \subset \mathbb{Z}^2$ is also a non-degenerate circuit of cardinality 3 and the same $b \in \mathbb{Z}^3$ from Example 2.3 works for this example as well. We then get exactly the same criteria for $c_1 + c_2 x_1^2 x_2^2 + c_3 x_1^7 x_2^7$ to have a degenerate root as in Example 2.3. However, $5 - 7x_1^2 x_2^2 + 2x_1^7 x_2^7$ has infinitely many degenerate roots in $\mathbb{R}_+^2$: They are all of the form $(x_1, x_2) = (r, 1/r)$ for $r \in \mathbb{R}_+$.* ⋄

We let $\mathrm{Conv}A$ denote the *convex hull* of A, i.e., the smallest convex set containing A.

Theorem 2.5. *[BRS09, Thm. 2.17] Following the notation of Theorem 2.2, $Z_+(f)$ is empty if and only if at least one of the following two conditions holds:*

1. *All the c_i have the same sign.*
2. *$\mathrm{Conv}A$ is a simplex, $\mathrm{sign}(b_1 c_1) = \cdots = \mathrm{sign}(b_{m+2} c_{m+2})$, and $(\Xi_A(f) + 1)^{\mathrm{sign}(b_j)} < 1$ where j is the unique index with $\mathrm{sign}(b_i b_j) < 0$ for all $i \neq j$.*

Furthermore, $Z_+(f)$ consists of a single point if and only if the following conditions all hold: $m = n$, $\mathrm{Conv}A$ is a simplex, $\mathrm{sign}(b_1 c_1) = \cdots = \mathrm{sign}(b_{m+2} c_{m+2})$, and $\Xi_A(f) = 0$. ■

Remark 2.6. *Unravelling the characterization above, we see that unless all the c_i have the same sign, and $\mathrm{Conv}A$ has a particular shape, we will need to compare a high-degree monomial in the c_i against 1 to know if $Z_+(f)$ is empty. The latter calculation is then clearly equivalent to computing the sign of $b_1 \log|c_1/b_1| + \cdots + b_{m+2} \log|c_{m+2}/b_{m+2}|$. This is our central reduction to linear forms in logarithms.* ⋄

Example 2.7. *Suppose $A \subset \mathbb{Z}^3$ consists of the columns of $\begin{bmatrix} 24 & 68 & -47 & 52 & 71 \\ -85 & -10 & -51 & 11 & 87 \\ -90 & 33 & 1 & 28 & 46 \end{bmatrix}$. Then $\mathrm{Conv}A$ is a simplex, and Theorem 2.5 (along with a bit of Morse Theory [BDPRRR24]) tells us (assuming $c_1, c_2, c_3, c_5 > 0 > c_4$) that*

$$Z_+\left(c_1 x_1^{24} x_2^{-85} x_3^{-90} + c_2 x_1^{68} x_2^{-10} x_3^{33} + c_3 x_1^{-47} x_2^{-51} x_3 + c_4 x_1^{52} x_2^{11} x_3^{28} + c_5 x_1^{71} x_2^{87} x_3^{46}\right)$$

is empty, a single point, or isotopic to a 2-sphere, according as $\sum_{i=1}^5 b_i \log|c_i|$ is less than, equal to, or greater than $\sum_{i=1}^5 b_i \log|b_i|$, where $b = (43403, 600796, 150818, -1138887, 343870)^\top$. This condition can clearly be handled reasonably via floating calculation on a computer — provided sufficient accuracy is used for the underlying logarithms. ⋄

Remark 2.8. *We thus see that the sign of $\Xi_A(f)$ (or, equivalently, the sign of $\log(\Xi_A(f) + 1)$) appears to determine the isotopy type of $Z_+(f)$, at least in certain orthants of coefficient space. We call the connected components of the complement of the zero set of $\Xi_A(f)$, in the orthants of $(\mathbb{R} \setminus \{0\})^{n+2}$,* discriminant chambers. *One aspect of circuits that helps make computing the isotopy type of $Z_+(f)$ tractable (for f a circuit polynomial) is that every orthant of $\mathbb{R}^{n+2}$ contains at most 2 discriminant chambers. So, in the circuit case, the topological behavior of $Z_+(f)$ depends mainly on whether $f \in Z_{\mathbb{R}}(\Xi_A)$, or on which "side" of $Z_{\mathbb{R}}(\Xi_A)$ f lies.* ⋄

The degenerate circuit analogue of Theorem 2.5 is similar. In particular, recall that for any degenerate circuit $A=\{a_1,\ldots,a_{m+2}\}\subset\mathbb{R}^n$ of cardinality $m+2$, with corresponding right null vector b for $\widehat{\mathcal{A}}$, the subset $B:=\{a_i \mid b_i\neq 0\}$ is a non-degenerate circuit. We also let $f_B(x):=\sum_{a_i\in B} c_i x^{a_i}$.

Theorem 2.9. *[BRS09, Thm. 2.18] Following the notation above, suppose A is a degenerate circuit, $f\in\mathbb{R}\left[x_1^{\pm 1},\ldots,x_n^{\pm 1}\right]$ is supported on A, and $f(x)=c_1x^{a_1}+\cdots+c_{m+2}x^{a_{m+2}}$. Then $Z_+(f)$ is empty if and only if at least one of the following conditions holds:*

1. *All the c_i have the same sign.*
2. *(a) $\mathrm{Conv}A$ is a simplex, (b) $\mathrm{sign}(b_ic_i)$ is constant as i ranges over all indices with $a_i\in B$, (c) $\mathrm{sign}(c_ic_j)<0$ for some i with $a_i\notin B$, and (d) $(\Xi_B(f_B)+1)^{\mathrm{sign}(b_j)}\leq 1$ where j is the unique index with $b_j\neq 0$ and $\mathrm{sign}(b_ib_j)\leq 0$ for all $i\neq j$.* ■

Example 2.10. *With $A=\{(0,0),(1,0),(2,0),(0,1)\}$ it is easily checked that $b=(1,-2,1,0)^\top$ is a suitable right nullvector for $\widehat{\mathcal{A}}$, and the j from Theorem 2.9 is $j=2$. So, for $f(x_1,x_2)=1-cx_1+x_1^2+x_2$ and $c>0$, we see that $f_B=1-cx_1+x_1^2$, $\Xi_B(f_B)+1=\frac{2}{c}$, $\mathrm{sign}(b_j)=1$, and thus $Z_+(f)$ is empty if and only if $c\leq 2$.* ⋄

So in the end, although the indexing is slightly more complicated, we can again reduce detecting emptiness of $Z_+(f)$ to checking the sign of an integer linear form in logarithms of integers.

Before moving on, we must also recall an explicit bound on the complexity of computing the sign of a linear form in logarithms. First, we recall the following paraphrase of a bound of Matveev [Mat00, Cor. 2.3], considerably strengthening earlier bounds of Baker and Wustholtz [BW93]. (See also [BMS06, Thm. 9.4].)

Theorem 2.11. *Suppose K is a degree d real algebraic extension of $\mathbb{Q}$, $c_1,\ldots,c_m\in K\setminus\{0\}$, and $b_1,\ldots,b_m\in\mathbb{Z}\setminus\{0\}$. Let $B:=\max\{|b_1|,\ldots,|b_m|\}$ and $\log H_i := \max\{dh(c_i),|\log c_i|,0.16\}$ for all i. Then $\sum_{i=1}^m b_i\log c_i\neq 0$ implies that $\log\left|\sum_{i=1}^m b_i\log c_i\right|$ is strictly greater than $-1.4\cdot m^{4.5}30^{m+3}d^2(1+\log d)(1+\log B)\prod_{i=1}^m\log H_i$.* ■

We must also recall the following classical fact on approximating logarithms via Arithmetic-Geometric Iteration:

Theorem 2.12. *[Ber03, Sec. 5] Given any positive $x\in\mathbb{Q}$ of logarithmic height h, and $\ell\in\mathbb{N}$ with $\ell\geq h$, we can compute $\lfloor\log_2\max\{1,\log|x|\}\rfloor$ and the ℓ most significant bits of $\log x$ in time $O(\ell\log^2\ell)$.* ■

Taking $d=1$, an immediate consequence of the preceding two bounds is the following algorithmic complexity bound:

Corollary 2.13. *[Roj22, Proof of Lemma 4.2] We can compute the sign of $\Lambda(b,c)$ in time $O((31\log H)^m\log(B)\log^2(\log(B)\log H)^2)$.* ■

By combining Corollary 2.13 with Theorems 2.5 and 2.9, we immediately obtain an explicit (deterministic) complexity bound for detecting positive roots for circuit polynomials, i.e., the main results of [BRS09]. However, the resulting complexity bound is exponential in n. Our entire goal is to reduce this time bound to polynomial in n and, thanks to our probabilistic corollaries, we'll at least accomplish this for a large fraction of inputs. But first let us complete our background by reviewing real root detection for a single $(n+1)$-nomial.

2.1 A Brief Note on the Case $k=n+1$

We mentioned earlier that detecting real roots for an n-variate $(n+1)$-nomial is much easier than for a $(n+2)$-nomial. This is because of the following folkloric lemma:

Lemma 2.14. *Suppose $f\in\mathbb{R}\left[x_1^{\pm 1},\ldots,x_n^{\pm 1}\right]$ can be written in the form $f(x)=\sum_{i=1}^{n+1}c_ix^{a_i}$ where $A=\{a_1,\ldots,a_{n+1}\}$ is the vertex set of an n-simplex, i.e., the rank of $\widehat{\mathcal{A}}$ is $n+1$. Then $Z_+(f)$ is empty if and only if all the c_i have the same sign.*

Proof: Substituting $x_i=e^{y_i}$ for all i, we see that $Z_+(f)$ is empty if and only if the real zero set, $Z_{\mathbb{R}}(g)$, of the exponential sum $g(y):=\sum_{j=1}^{n+1}c_je^{a_j\cdot y}$ is empty. Since $Z_{\mathbb{R}}(g)$ is invariant under translation of A, we may assume a_1 is the origin.

Noting that the emptiness of $Z_{\mathbb{R}}(g)$ is invariant under invertible linear maps applied to the variables, we can substitute $y\mapsto My$, where we can consider y as a column vector, and let M be the inverse of the $n\times n$ matrix whose ith column is a_{i+1}. (M is invertible since the edge vectors of any vertex of a simplex are linearly independent.) So we may assume $g(y)=c_1+c_2e^{y_1}+\cdots+c_{n+1}e^{y_n}$. Finally, since $Z_{\mathbb{R}}(g)$ is invariant under nonzero scaling of g, and the emptiness of $Z_{\mathbb{R}}(g)$ is invariant under translation of the variables, we may assume $g(y)=\varepsilon_1+\varepsilon_2e^{y_1}+\cdots+\varepsilon_{n+1}e^{y_n}$ where $\varepsilon_i\in\{\pm1\}$ has the same sign as c_i. Letting $u_i=e^{y_i}$ for all i, we are reduced to deciding the emptiness of $Z_+(\varepsilon_1+\varepsilon_2u_1+\cdots+\varepsilon_{n+1}u_n)$. The latter zero set is clearly empty if and only if all the ε_i have the same sign. ■

3 THE PROOF OF THEOREM 1.4

First note that, out of the 2^{n+2} orthants of $(c_1,\ldots,c_{n+2})\in(\mathbb{R}\setminus\{0\})^{n+2}$, exactly *two* of these orthants satisfy the condition

$$(\star)\qquad \mathrm{sign}(b_1c_1)=\cdots=\mathrm{sign}(b_{n+2}c_{n+2}).$$

For those orthants *not* satisfying Condition ($\star$), Theorems 2.5 and 2.9 tell us that checking $Z_+(f)\stackrel{?}{=}\emptyset$ is almost trivial: We merely need to check whether all the c_i have the same sign. Note also that the sign of $\Xi_A(f)$ (or, equivalently, $\log(\Xi_A(f)+1)$) is independent of the signs of the c_i. So the inputs where checking $Z_+(f)\stackrel{?}{=}\emptyset$ is harder are exactly the inputs where $\log(\Xi_A(f)+1)$ requires more accuracy to evaluate. So by Corollary 1.10, we obtain that we can decide $Z_+(f)\stackrel{?}{=}\emptyset$ easily on a fraction of $1-O\left(\frac{1}{2^nH}\right)$ of our input f, since our underlying probability measure is uniform across all orthants.

So now we must precisely quantify what we mean by "more accuracy" and "easily": Corollary 1.10 tells us that in the two orthants satisfying Condition ($\star$), with probability $1-O(1/H)$, we have:

$$\left|\left(\sum_{i=1}^{n+2}b_i\log|c_i|\right)-\left(\sum_{i=1}^{n+2}b_i\log|b_i|\right)\right| \quad > \quad \frac{\sqrt{n+2}}{H} \tag{3}$$

if

$$\sum_{i=1}^{n_2}b_i\log|c_i|\neq\sum_{i=1}^{n_2}b_i\log|b_i|.$$

In other words, we now know that for most inputs in our two special orthants, "moderate" accuracy for each logarithm in the sums above will suffice to correctly determine which of $\sum_{i=1}^{n_2}b_i\log|c_i|$ or

$\sum_{i=1}^{n_2} b_i \log|b_i|$ is bigger (or if they are equal). More precisely, simply let $B := \max_i |b_i|$ and let L_i and M_i be rational numbers satisfying $|L_i - \log|c_i|| < \frac{\sqrt{n+2}}{6nBH}$ and $|M_i - \log|b_i|| < \frac{\sqrt{n+2}}{6nBH}$. Then by the Triangle Inequality, the values of $\left(\sum_{i=1}^{n+2} b_i \log|c_i|\right) - \left(\sum_{i=1}^{n+2} b_i \log|b_i|\right)$ and $\left(\sum_{i=1}^{n+2} L_i\right) - \left(\sum_{i=1}^{n+2} M_i\right)$ differ by no more than $\frac{\sqrt{n+2}}{3H}$. In other words, to decide whether $\Lambda(b,c)$ is negative, zero, or positive, we merely check whether $\left(\sum_{i=1}^{n+2} L_i\right) - \left(\sum_{i=1}^{n+2} M_i\right)$ is less than $-\frac{2\sqrt{n+2}}{3H}$, inside of the open interval $\left(-\frac{1\sqrt{n+2}}{3H}, \frac{1\sqrt{n+2}}{3H}\right)$, or greater than $\frac{2\sqrt{n+2}}{3H}$: These are the only possibilities that can occur on our $1 - O(1/H)$ fraction of inputs from our two orthants satisfying Condition ($\star$), thanks to Corollary 1.10.

To conclude, observe that Cramer's Rule (and Hadamard's Inequality for determinants) tells us that the height of b_i is $O(n\log(dn))$. So by Theorem 2.12, each $\log|c_i|$ and $\log|b_i|$ term can be approximated to our desired accuracy in time

$$O((n + \log(dH) + \log(H) + n\log(dn))\log^2(n\log(ndH))),$$

which is simply $O(n\log^3(ndH))$. So computing $L_1, M_1, \ldots, L_{n+2}, M_{n+2}$ takes time $O(n^2\log^3(ndH))$. The computation of b takes time $n^{3.373}\log^{1+o(1)}(nd)$ via fast integer linear algebra (see, e.g., [Roj22, Lemma 2.1]). So our overall time bound is

$O(n^2\log^3(ndH)) + n^{3.373}\log^{1+o(1)}(nd) = O(n^{3.373}\log^3(ndH))$. ■

Example 3.1. *Suppose $A \subset \mathbb{Z}^5$ consists of the columns of*

$$\begin{bmatrix} -48 & -70 & -31 & -41 & 86 & -44 & 10 \\ -13 & -87 & 53 & -93 & -82 & 68 & -79 \\ 36 & -75 & -78 & -75 & -59 & -91 & 54 \\ 76 & 95 & -22 & -93 & 68 & 30 & -86 \\ -46 & 96 & 47 & -11 & 54 & 21 & 54 \end{bmatrix}.$$

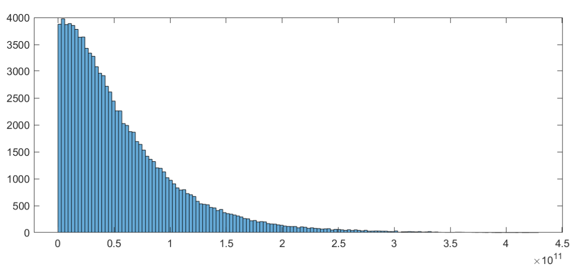
Then a suitable b vector is

$(13114054985, 1804628444, 48927499024, 2016784302, 2855329886, -51793775050, -16924521591)$.

More to the point, let us consider the distribution of the values of $\log(\Xi_A(f)+1)$ as the coefficients of f range uniformly over $\{\pm1, \ldots, \pm1000\}$: After a sample of 10^7 random trials, we found a minimal value of 8498.1 for $\log(\Xi_A(f)+1)$, attained at

$$(c_1, \ldots, c_7) = (996, 938, 176, 703, 431, -783, -44).$$

A histogram for the values of $\log(\Xi_A(f)+1)$ from our trial is plotted below:



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