Counting Real Connected Components of Trinomial Curve Intersections and m-nomial Hypersurfaces[0]

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In memory of Konstantin Alexandrovich Sevast'yanov, 1956–1984.

Abstract

We prove that any pair of bivariate trinomials has at most 5 isolated roots in the positive quadrant. The best previous upper bounds independent of the polynomial degrees were much larger, e.g., 248832 (for just the non-degenerate roots) via a famous general result of Khovanski. Our bound is sharp, allows real exponents, allows degeneracies, and extends to certain systems of n-variate fewnomials, giving improvements over earlier bounds by a factor exponential in the number of monomials. We also derive analogous sharpened bounds on the number of connected components of the real zero set of a single n-variate m-nomial.

1 Introduction

Generalizing Descartes' Rule of Signs to multivariate systems of polynomial equations has proven to be a significant challenge. Recall that a weak version of this famous classical result asserts that any real univariate polynomial with exactly m monomial terms has at most $m-1$ positive roots. This bound is sharp and generalizes easily to real exponents (cf. Section 2). The original statement in René Descartes' *La Géométrie* goes back to June of 1637 and Latham and Smith's English translation states that this result was observed even earlier by Thomas Harriot in his *Artis Analyticae Praxis* (London, 1631) [SL54, Footnote 196, Pg. 160]. Proofs can be traced back to work of Gauss around 1828 and other authors earlier, but a definitive sharp bound for multivariate polynomial systems seems to have elluded us in the second millenium. This is particularly unfortunate since systems of sparse polynomial equations, and inequalities, now occur in applications as diverse as radar imaging [FH95], chemistry [GH99], and neural net learning [VR02].

Here we take another step toward a sharp, higher-dimensional generalization of Descartes' bound by providing the first significant improvement on the case of curves in the plane, and certain higher-dimensional cases, since Khovanski's seminal work in the early 1980's [Kho80]. Khovanski's revolutionary **Theory of Fewnomials** [Kho80, Kho91] extends Descartes' bound to a broader class of analytic functions (incorporating certain measures of "input complexity") as well as higher dimensions, but the resulting bounds are impractically large even in the case of two variables. Our bounds are sharper than Khovanski's by a factor exponential in the number of monomial terms, allow degeneracies, and are optimal for the case of two bivariate trinomials. We then present similar

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sharpenings, also allowing real exponents and degeneracies, for the number of compact and non-compact connected components of the real zero set of a single sparse polynomial with any number of variables.

1.1 Main Results

Perhaps the simplest way to generalize the setting of Descartes' Rule to higher dimensions and real exponents is the following:

Notation. *For any* $c \in \mathbb{R}^* := \mathbb{R}\setminus\{0\}$ *and* $a=(a_1,\ldots,a_n)\in\mathbb{R}^n$*, let* $x^a := x_1^{a_1}\cdots x_n^{a_n}$ *and call* cx^a *a* **monomial term**. *We will refer to* $\mathbb{R}^n_+ := \{x\in\mathbb{R}^n \mid x_i>0 \text{ for all } i\}$ *as the* **positive orthant**. *Henceforth, we will assume that* $F:=(f_1,\ldots,f_k)$ *where, for all* i*,* $f_i\in\mathbb{R}[x^a \mid a\in\mathbb{R}^n]$ *and* f_i *has exactly* m_i *monomial terms. We call* f_i *an* $\boldsymbol{n}$**-variate** $\boldsymbol{m_i}$**-nomial** *and, when* $m_1,\ldots,m_k\geq 1$*, we call* F *a* $\boldsymbol{k\times n}$ **fewnomial system** *(over* $\mathbb{R}$*)* **of type** $\boldsymbol{(m_1,\ldots,m_k)}$. *We call any homeomorphic image of the unit circle or a (closed, open, or half-open) interval an* **arc**. *Finally, we say a real root* ζ *of* F *is* **isolated** *(resp.* **smooth**, **non-degenerate**, *or* **non-singular***) iff the only arc of real roots of* F *containing* ζ *is* ζ *itself (resp. the Jacobian of* F*, evaluated at* ζ*, has full rank).* $\diamond$

Definition 1. *For any* $m_1,\ldots,m_n\in\mathbb{N}$*, let* $\mathcal{N}'(m_1,\ldots,m_n)$ *(resp.* $\mathcal{N}(m_1,\ldots,m_n)$*) denote the maximum number of non-degenerate (resp. isolated) roots an* $n\times n$ *fewnomial system of type* $(m_1,\ldots,m_n)$ *can have in the positive orthant.* $\diamond$

Finding a tight upper bound on $\mathcal{N}(m_1,\ldots,m_n)$ for $n\geq 2$ remains a central problem in real algebraic geometry which is still poorly understood. For example, Anatoly Georgievich Kushnirenko conjectured in the mid-1970's [Kho80] that $\mathcal{N}'(m_1,\ldots,m_n)=\prod_{i=1}^n(m_i-1)$, at least for the case of integral exponents. The $n\times n$ polynomial system

$$\left(\prod_{i=1}^{m_1-1}(x_1-i),\ldots,\prod_{i=1}^{m_n-1}(x_n-i)\right) \tag{1}$$

was already known to provide an easy lower bound of $\mathcal{N}'(m_1,\ldots,m_n)\geq\prod_{i=1}^n(m_i-1)$, but almost 30 years would pass until a counter-example to Kushnirenko's conjecture was published [Haa02] (see Formula (5) in Section 1.2). However, it was known much earlier that Kushnirenko's conjectured upper bound could not be extended to $\mathcal{N}(m_1,\ldots,m_n)$: The trivariate polynomial system

$$\left(x(z-1), y(z-1), \prod_{i=1}^{5}(x-i)^2+\prod_{i=1}^{5}(y-i)^2\right) \tag{2}$$

is of type $(2,2,21)$, has exactly 25 ($>20=1\cdot1\cdot20$) roots in the positive octant, all of which are isolated and integral, but with Jacobian of rank <3 (see, e.g., [Stu98, note added in proof] or [Ful84, Ex. 13.6, Pg. 239]). Indeed, $\mathcal{N}'(m_1,\ldots,m_n)\leq\mathcal{N}(m_1,\ldots,m_n)$ for all $m_1,\ldots,m_n$, since a non-degenerate non-isolated root in $\mathbb{R}^n$ can never have more than $n-1$ tangent planes with linearly independent normal vectors. Cases where the inequality $\mathcal{N}'(m_1,\ldots,m_n)\leq\mathcal{N}(m_1,\ldots,m_n)$ is strict appear to be unknown.

Interestingly, allowing degeneracies and real exponents introduces more flexibility than trouble in our approach: The proof of our first main result is surprisingly elementary, using little more than exponential coordinates and an extension of Rolle's Theorem from calculus.

Definition 2. *For any* $S\subseteq\mathbb{R}^n$*, let* $\mathrm{Conv}(S)$ *denote the smallest convex set containing* S*. Also, for any* m*-nomial of the form* $f(x):=\sum_{a\in A}c_a x^a$*, we call* $\mathrm{Supp}(f):=\{a \mid c_a\neq 0\}$ *the* **support** *of* f*, and define* $\mathrm{Newt}(f):=\mathrm{Conv}(\mathrm{Supp}(f))$ *to be the* **Newton polytope** *of* f*. Finally, we let* $Z_+(F)$ *denote the zero set of* F *in* $\mathbb{R}^n_+$. $\diamond$

Theorem 1. *We have* $\mathcal{N}'(3,3)=\mathcal{N}(3,3)=5$ *and, more generally:*

(a) $\mathcal{N}'(3,m)=\mathcal{N}(3,m)\leq 2^m-2$ *for all* $m\geq 4$.

(b) *Any $n \times n$ fewnomial system $F := (f_1, \ldots, f_n)$ of type $(m_1, \ldots, m_{n-1}, m)$ with*

- **(i)** $b_1 + \mathrm{Supp}(f_1), \ldots, b_{n-1} + \mathrm{Supp}(f_{n-1}) \subseteq A$ *for some* $b_1, \ldots, b_{n-1} \in \mathbb{R}^n$ *and* $A \subset \mathbb{R}^n$ *of cardinality* $n+1$
- **(ii)** $Z_+(f_1, \ldots, f_{n-1})$ *smooth*

has no more than $n + n^2 + \cdots + n^{m-1}$ isolated roots in $\mathbb{R}^n_+$, for all $m, n \geq 1$. Also, for such F, the maximum number of non-degenerate and isolated roots in $\mathbb{R}^n_+$ are equal.

(c) *For any $\alpha_1, \alpha_2, \alpha_3, a_2, b_2, c_3, d_3, r_1, s_1, u_2, v_2 \in \mathbb{R}$ and any degree D polynomial $p \in \mathbb{R}[S_1, S_2]$ with $Z_+(p)$ smooth, the 2×2 fewnomial system*

$$(\star) \begin{cases} \alpha_1 + \alpha_2 x^{a_2} y^{b_2} + \alpha_3 x^{c_3} y^{d_3} \\ p(x^{r_1} y^{s_1}, x^{u_2} y^{v_2}) \end{cases}$$

has no more than $4\mathrm{Area}(\mathrm{Newt}(p)) + 2D + 1$ ($\leq 6D+1$) isolated roots in $\mathbb{R}^2_+$, where we normalize area so that the unit square has area 2.

The quantities $\mathcal{N}'(m_1, \ldots, m_n)$ and $\mathcal{N}(m_1, \ldots, m_n)$ are much easier to compute when some m_i is bounded above by 2: all families of polynomial systems currently known to admit explicit formulae, including $n \times n$ **binomial** systems, are summarized in Theorem 4 of Section 2.

Remark 1. *The value of $\mathcal{N}(3,3)$ was previously unknown and there appears to be no earlier result directly implying the equality $\mathcal{N}'(3,m) = \mathcal{N}(3,m)$ for any m. In particular, the only other upper bound on $\mathcal{N}'(3,m)$ or $\mathcal{N}(3,m)$ until now was $\mathcal{N}'(3,m) \leq 3^{m+2} 2^{(m+2)(m+1)/2}$, which evaluates to 248832 when $m=3$. The best previous bound for the systems described in (b) and (c) were respectively $(n+1)^{m+n} 2^{(m+n)(m+n-1)/2}$ and $1024D(D+2)^5$, counting only the non-degenerate roots. (See Khovanski's Theorem on Real Fewnomials in Section 1.2 and Proposition 1 of Section 2.)* $\diamond$

Example 1. *Note that while we still don't know an upper bound on $\mathcal{N}'(4,4)$ better than Khovanski's 4586471424, we at least obtain a new approach for certain fewnomial systems with many monomial terms. For instance, part (c) of our first main theorem tells us that the 2×2 fewnomial system:*

$$\alpha_1 + \alpha_2 x^{a_2} y^{b_2} + \alpha_3 x^{c_3} y^{d_3}$$

$$\beta_0 + \beta_{-1} x^{r_1} y^{s_1} + \beta_1 x^{r_1+u_2} y^{s_1+v_2} + \beta_2 x^{2(r_1+u_2)} y^{2(s_1+v_2)} + \cdots + \beta_{100} x^{100(r_1+u_2)} y^{100(s_1+v_2)}$$

has no more than $801 = 4 \cdot 100 + 2 \cdot 200 + 1$ isolated roots in $\mathbb{R}^2_+$, for all $\alpha_i, a_2, b_2, c_3, d_3, r_1, s_1, u_2, v_2 \in \mathbb{R}$ and $\beta_i \in \mathbb{R}$ such that $\beta_0 + \beta_1 u + \cdots + \beta_{100} u^{100}$ has no degenerate roots. The pair of fewnomials $\left((y-1)(y-2), y - \prod_{j=1}^{100}(x-j)\right)$ easily shows us that a system in this family can have as many as 200 non-degenerate roots, if not more. Khovanski's Theorem on Real Fewnomials below yields an upper bound of 68878994643353600 for just the number of non-degenerate roots. $\diamond$

We can also classify when a pair of bivariate trinomials has 5 isolated roots in the positive quadrant via Newton polygons. In particular, note that while one can naturally associate a pair of polygons to F when $n=2$, we can also associate a **single** polygon by forming the **Minkowski sum** $P_F := \mathrm{Newt}(f_1) + \mathrm{Newt}(f_2)$. We can then give the following addendum to Theorem 1.

Corollary 1. *A 2×2 fewnomial system F of type $(3,3)$ respectively has at most 0, 2, or 4 isolated roots in $\mathbb{R}^2_+$, according as we restrict to those F with P_F a line segment, triangle, or ℓ-gon with $\ell \in \{4,5\}$.*

The central observation that led to Theorem 1 may be of independent interest. We state it as assertion (5) of Theorem 2 below. However, let us first define two more combinatorial quantities closely related to $\mathcal{N}'$ and $\mathcal{N}$.

Definition 3. *For any $\mu, n \in \mathbb{N}$, we say that a $k \times n$ fewnomial system with exactly μ distinct exponent vectors is **μ-sparse**. Also, let $\mathcal{K}'(n,\mu)$ (resp. $\mathcal{K}(n,\mu)$) denote the maximum number of non-degenerate (resp. isolated) roots a μ-sparse $n \times n$ fewnomial system can have in the positive orthant.* $\diamond$

Assertions (2) and (3) of our next main result dramatically refine the bounds of Oleinik, Petrovsky, Milnor, Thom, and Basu on the number of connected components of a real algebraic set [OP49, Mil64, Tho65, Bas99] in the special case of a single polynomial, and hold in the more general context of real exponents:

Theorem 2. *Let f be any n-variate m-nomial and let $P(n,m)$ denote be the maximum number of connected components of $Z_+(f)$ over all n-variate m-nomials. Also let $P_{\mathrm{comp}}(n,m)$ (resp. $P_{\mathrm{non}}(n,m)$) be the corresponding quantity counting just the compact (resp. non-compact) connected components. Finally, for any $r_1,s_1,u_2,v_2\in\mathbb{R}$ and any degree D polynomial $p\in\mathbb{R}[S_1,S_2]$, let $\rho(x,y):=p(x^{r_1}y^{s_1},x^{u_2}y^{v_2})$. Then we have:*

0. *$P_{\mathrm{comp}}(n+1,2)=0$, $P_{\mathrm{non}}(n+1,2)=1$, $P_{\mathrm{comp}}(1,m)=m-1$, $P_{\mathrm{non}}(1,m)=0$, and $P_{\mathrm{non}}(n,0)=1$ for all $m,n\geq 1$.*
1. *$P_{\mathrm{comp}}(n,m)=0$ and $P_{\mathrm{non}}(n,m)=P_{\mathrm{non}}(m-1,m)\leq P(m-2,m)$ for $3\leq m\leq n+1$.*
2. *$\max\left\{\left\lfloor\frac{m}{2}\right\rfloor-n-1,\left(\left\lfloor\frac{m-1}{2n}\right\rfloor-1\right)^n\right\}\leq P_{\mathrm{comp}}(n,m)\leq 2\lfloor\mathcal{K}'(n,m)/2\rfloor$ for all $n\geq 2$, and the last multiple of 2 can be removed in the smooth case. Also, $Z_+(\rho)$ has no more than $\mathrm{Area}(\mathrm{Newt}(p))$ compact components, where we normalize area so that the unit square has area 2.*
3. *$\max\left\{m-1,\left(\left\lfloor\frac{m-1}{2(n-1)}\right\rfloor-1\right)^{n-1}\right\}\leq P_{\mathrm{non}}(n,m)\leq 2P(n-1,m)$ for all $n\geq 2$. Also, $Z_+(\rho)$ has no more than $2D$ non-compact connected components.*

Furthermore, in the special case where $n=2$ and $Z_+(f)$ is smooth, let $I(m)$ (resp. $V(m)$) denote maximum number of isolated[1] *inflection points (resp. isolated points of vertical tangency) of $Z_+(f)$. Then we also have*

4. *$V(m)\leq\mathcal{K}(2,m)$ for all $m\geq 1$, and $Z_+(\rho)$ has no more than $\mathrm{Area}(\mathrm{Newt}(p))$ isolated points of vertical tangency.*
5. *$I(m)\leq 3\mathcal{K}'(2,m)$ for all $m\leq 3$, and $Z_+(\rho)$ has no more than $3\mathrm{Area}(\mathrm{Newt}(p))$ isolated inflection points.*

In particular, $V(3)\leq 1$ and $I(3)\leq 3$, even if $Z_+(f)$ is not smooth.

Note that a non-compact connected component of $Z_+(f)$ can still have compact closure, since $\mathbb{R}^n_+$ is not closed in $\mathbb{R}^n$, e.g., $\left\{(x_1,x_2)\in\mathbb{R}^2_+ \mid x_1^2+x_2^2=1\right\}$.

While the above bounds on the number of connected components are non-explicit, they are stated so they can immediately incorporate any advance in computing $\mathcal{K}'(n,\mu)$. So for a general and explicit upper bound independent of the underlying polynomial degrees now, one could, for instance, simply insert the explicit upper bound for $\mathcal{K}'(n,\mu)$ appearing in Khovanski's Theorem on Real Fewnomials (see Section 1.2 below) and the formula $\mathcal{K}'(2,4)=5$ implied by Theorem 1 (see Proposition 1 of Section 2).

Corollary 2. *Following the notation of Theorem 2, $P(n,m)\leq\mathcal{K}'(n,m)+2P(n-1,m)$ for all $n\geq 2$. More explicitly, $P(n,m)\leq\sum_{i=0}^{n-1}2^i\mathcal{K}(n-i,m)\leq n(n+1)^m 2^{n-1}2^{m(m-1)/2}$. In particular, a tetranomial curve in $\mathbb{R}^2_+$ has no more than 4 compact connected components and no more than*[2] *4 non-compact connected components.* ■

The bound above is already significantly sharper than an earlier bound of $(2n^2-n+1)^m(2n)^{n-1}2^{m(m-1)/2}$, which held only for the smooth case, following from [Kho91, Sec. 3.14, Cor. 5]. The bounds of Theorem 2 also improve an earlier result of the middle author on smooth algebraic hypersurfaces [Roj00a, Cor. 3.1].

Our final main result shows us that we can considerably refine assertion (3) of Theorem 2 if we take advantage of the underlying polyhedral structure.

[1] relative to the locus of points of inflection (resp. points of vertical tangency)...

[2] Theorem 2 actually yields an upper bound of 6 for the number of non-compact components so we cheated slightly by using Theorem 3 below to get the very last bound.

Definition 4. *For any* $w:=(w_1,\ldots,w_n)\in\mathbb{R}^n$ *and any compact set* $B\subset\mathbb{R}^n$*, we let* B^w *— the* **face of** $\boldsymbol{B}$ **with inner normal** $\boldsymbol{w}$ *— be the set of all* $x\in B$ *minimizing the inner product* $w\cdot x$*. Finally, for any any* n*-variate* m*-nomial* f *of the form* $\sum_{a\in A}c_ax^a$*, we let* $\mathrm{Init}_w(f)$ *— the* **initial terms** *of* f *(with respect to* w*) — be* $\sum_{a\in A^w}c_ax^a$*.* $\diamond$

Recall that the dimension of a polytope $P\subseteq\mathbb{R}^n$ is the dimension of the smallest subspace containing a translate of P and that a **facet** of an n-dimensional polytope is simply a face of dimension $n-1$.

Theorem 3. *Let* f *be any* n*-variate* m*-nomial* f *with* n*-dimensional Newton polytope. Assume further that* $Z_+(\mathrm{Init}_w(f))$ *is smooth for all* $w\in\mathbb{R}^n\setminus\{\mathbf{O}\}$*. Then the number of non-compact connected components of* $Z_+(f)$ *is no more than*

$$\sum_{\substack{w\text{ a unit inner facet}\\ \text{normal of }\mathrm{Newt}(f)}} N_w,$$

where N_w *denotes the number of the number of connected components of* $Z_+(\mathrm{Init}_w(f))$*. In particular, this bound is no larger than* $\sum\limits_{Q\text{ a facet of }P} P(n-1,\#\mathrm{Supp}(f)\cap Q)$*. Finally, when* $n=2$ *and* $Z_+(f)$ *is smooth as well, the last upper bound is sharp and can be simplified to* $\lfloor m'/2\rfloor$*, where* m' *is the number of points of* $\mathrm{Supp}(f)$ *lying on the boundary of* $\mathrm{Newt}(f)$*.*

Note that $Z_+(f)$ need not be smooth and our bound above is completely independent of the number of exponent vectors lying in the interior of $\mathrm{Newt}(f)$. The bivariate example $f(x,y)=y-\prod_{i=1}^{m'-2}(x-i)$ easily shows that the very last bound is sharp. A more intricate trivariate example follows.

Example 2. *Taking* $n=3$*, suppose* f *is*

$$\alpha_1+\alpha_2x^{3a}+\alpha_3z^{3c}+\alpha_4x^{3a}z^{3c}+\beta_1x^ay^bz^c+\beta_2x^{2a}y^bz^c+\beta_3x^ay^bz^{2c}+\beta_4x^{2c}y^bz^{2c}+\sum_{i=1}^K\gamma_ix^{a_i}y^{b_i}z^{c_i},$$

where K *is any positive integer, the* α_j*,* β_j*,* γ_j *are any nonzero real constants,* $a,b,c>0$ *and, for all* i*,* $a<a_i<2a$*,* $c<c_i<2c$*, and* $0<b_i<b$*. Note that* $\mathrm{Newt}(f)$ *is a snub pyramid with a rectangular base and thus has the same face lattice as a cube. Note also that no exponent vector of* f *lies in the relative interior of any face of* $\mathrm{Newt}(f)$ *of dimension* 1 *or* 2*. It is then easily checked that*

$$(\alpha_1\alpha_4^2-\alpha_2\alpha_3\alpha_4)(\beta_1\beta_4^2-\beta_2\beta_3\beta_4)Q(\alpha,\beta)\neq 0,$$

where Q *is a product of* 4 *more complicated polynomials, is a sufficient condition for all the* $Z_+(\mathrm{Init}_w(f))$ *to be smooth. So, under the last assumption, Theorem 3 tells us that the zero set of* f *in the positive octant has no more than* $6P(2,4)\leq 6\cdot(4+6)=60$ *non-compact connected components, employing corollary 2 and the obvious fact that* $P(n,m)\leq P_{\mathrm{comp}}(n,m)+P_{\mathrm{non}}(n,m)$ *for the first inequality. Note that Theorem 2 would have given us a less explicit upper bound of* $P_{\mathrm{non}}(3,8+K)$ *which, by assertion (3), exceeds* 60 *for all* $K\geq 37$ *(if not earlier).* $\diamond$

Note also that the assumption on the $\mathrm{Init}_w(f)$ is rather mild: it follows easily from Sard's Theorem [Hir94] that our smoothness condition will hold for a generic choice of the coefficients of f, e.g., all coefficient vectors outside a set of measure zero in $\mathbb{C}^{\#\mathrm{Supp}(f)}$ depending only on $\mathrm{Supp}(f)$. In particular, this hypothesis can become vacuous depending on the underlying Newton polytope.

Corollary 3. *Following the notation above, assume that* $\mathrm{Newt}(f)$ *is* **simplicial** *(i.e., for all* $d<\dim\mathrm{Newt}(f)$ *every* d*-dimensional face of* $\mathrm{Newt}(f)$ *has exactly* $d+1$ *vertices) and that [the relative interior of a face* Q *of* $\mathrm{Newt}(f)$ *contains a point of* $\mathrm{Supp}(f)\Longrightarrow Q$ *is a vertex]. Then* $Z_+(\mathrm{Init}_w(f))$ *is smooth for all* $w\in\mathbb{R}^n\setminus\{\mathbf{O}\}$*.* ■

Corollary 3 follows easily from the fact that for such an f, and any vector $w\in\mathbb{R}^n\setminus\{\mathbf{O}\}$, $Z_+(\mathrm{Init}_w(f))$ is analytically diffeomorphic to $\mathbb{R}_+^{n-1}$. The latter fact in turn follows easily via a monomial change of variables (cf. Proposition 2 of Section 2).

1.2 Important Related Results

The only available results for bounding the number of real roots, other than those coming from Fewnomial Theory [BC76, Gri82, Ris85, Kho91, Zel99, Roj00a], depend strongly on the individual exponents of F and are actually geared more toward counting complex roots, e.g., [BKK76, Kaz81, BLR91, Roj99]. (We also note that while the bounds of [Zel99] generalize Khovanski's theory to solution sets of **inequalities** involving Pfaffian functions, they do not appear to yield any new bounds on the quantities $\mathcal{K}$ and $\mathcal{N}$ we study.) So proving just $\mathcal{N}(3,3) < \infty$ already suggests an analytic approach. Nevertheless, the bounds from [BKK76, Kaz81, BLR91, Roj99] can be quite practical when the exponents are integral and the degrees of the polynomials are small.

Let us also point out that the term "fewnomial" is due to Kushnirenko and that the first explicit bounds in Fewnomial Theory were derived (not yet in complete generality) by Konstantin Alexandrovich Sevast'yanov in unpublished work around 1979 [Kho02]. Dima Yu. Grigoriev and Askold Georgevich Khovanski have also pointed out that shortly after Kushnirenko formulated his conjecture, a simple counter-example with $n=2$ was found by a student at Moscow State University [Gri00, Kho02]. Unfortunately, while the counter-example was verified by Khovanski himself [Kho02], it does not seem to have been recorded and the name of its inventor (who left mathematics immediately after graduating) seems to have been forgotten.

As for the size of upper bounds on the number of real roots, it is interesting to note that the best current general bounds independent of the polynomial degrees are exponential in the number of monomial terms of F, even for fixed n. Observe one of the masterpieces of real algebraic geometry.

Khovanski's Theorem on Real Fewnomials . *(See [Kho80] and [Kho91, Cor. 6, Pg. 80, Sec. 3.12].) We have* $\mathcal{K}'(n,\mu) \leq (n+1)^{\mu} 2^{\mu(\mu-1)/2}$. *More generally, the* $n \times n$ *fewnomial system*

$$q_1(x) = \cdots = q_n(x) = 0,$$

where each q_j *is a polynomial of degree* D_i *in* $x_1, \ldots, x_n$ *and* $x^{a_1}, \ldots, x^{a_\mu}$ *for some* $a_1, \ldots, a_\mu \in \mathbb{R}^n$, *has no more than* $2^{\mu(\mu-1)/2}(1+\sum_{i=1}^n D_i)^{\mu} \prod_{i=1}^n D_i$ *non-degenerate roots in* $\mathbb{R}^n_+$. ■

Finding non-trivial lower bounds on even $\mathcal{K}'(2,\mu)$ seems quite hard and surprisingly little else is known about what an optimal version of Khovanski's Theorem on Real Fewnomials should resemble. For example, around 1996, Ilya Itenberg and Marie-Françoise Roy proposed a conjectural polyhedral generalization of Descartes' Rule to multivariate systems of equations [IR96], based on a famous construction from Oleg Viro's 1983 Leningrad thesis (see, e.g., [Vir84]) and later extensions by Bernd Sturmfels [Stu94]. A bit later, Sturmfels offered US$500 for a proof that Itenberg and Roy's proposed upper bound held for the following family of 2×2 systems of type $(4,4)$:

$$(-x^5 + a_1 y^5 + a_2 x^3 y^5 + a_3 x^6 y^8, -y^5 + b_1 x^5 + b_2 x^5 y^3 + b_3 x^8 y^6), \tag{3}$$

where $a_1, a_2, a_3, b_1, b_2, b_3 > 0$. The Itenberg-Roy conjecture yields an alleged upper bound of 3 for this family, and Jeff Lagarias and Thomas Richardson later won Sturmfels' prize by showing that this bound in fact holds [LR97]. However, the Itenberg-Roy conjecture was later invalidated by the 2×2 system

$$(y - x - 1, y^3 + 0.01x^3 y^3 - 9x^3 - 2) \tag{4}$$

found by the left and right authors (Li and Wang): this system has exactly 3 roots in $\mathbb{R}^2_+$, whereas the conjectured bound would have only been 2 [LW98]. Perhaps the most important counter-example in this growing theory is Haas' recent counter-example to Kushnirenko's Conjecture: It is

$$(x_1^{108} + 1.1x_2^{54} - 1.1x_2, x_2^{108} + 1.1x_1^{54} - 1.1x_1), \tag{5}$$

which has 5 ($> 4 = 2 \cdot 2$) roots in the positive quadrant [Haa02]. Jan Verschelde has also verified numerically via his software package `PHCPACK` [Ver99] that there are exactly $108^2 = 11664$ complex roots, and thus (assuming the floating-point calculations were sufficiently good) each root is non-degenerate by Bézout's theorem on intersections of complex hypersurfaces [Sha77, ex. 1, pg. 198].

As for asymptotic behavior, it is still unknown whether even $\mathcal{K}'(2,\mu)$ is polynomial in μ: even the special case of 2×2 fewnomial systems of type $(3,m)$ is still open. Note also that this kind of polynomiality requires the number of variables to be fixed: the system $(x_1^2-3x_1+2,\ldots,x_n^2-3x_n+2)$ shows us that $\mathcal{N}'(\underbrace{3,\ldots,3}_{n})$ is already exponential in n. More to the point, it is also unknown whether a simple modification of Kushnirenko's conjectured bound (e.g., increasing the formula by a constant power or a factor exponential in n) would at last yield a true, sharp, and general improvement of Khovanski's Theorem on Real Fewnomials. The $2k\times 2k$ fewnomial system

$$(x_1^{108}+1.1y_1^{54}-1.1y_1, y_1^{108}+1.1x_1^{54}-1.1x_1,\ldots,x_k^{108}+1.1y_k^{54}-1.1y_k, y_k^{108}+1.1x_k^{54}-1.1x_k),$$

thanks to Haas' counter-example, easily shows that one needs at least an extra multiple no smaller than $\left(\frac{\sqrt{5}}{2}\right)^n$ if some Kushnirenko-like bound is to be salvaged.

Another question with even deeper implications is whether there is an algorithm for approximating the real roots of a fewnomial system whose complexity depends mainly on the number of **real** roots. Since all current algorithms for real-solving have complexity bounds essentially matching the analogous bounds for solving over the complex numbers, a positive answer would yield tremendous speed-ups, both practical and theoretical, for real-solving. However, little is known beyond the special cases of $n\times n$ binomial systems [Roj00a, Main Thm. 1.3] and univariate polynomials with 3 monomial terms or less [RY02]: For these cases, one can indeed obtain algorithms beating the known lower bounds [Ren89] for solving over the complex numbers, and [RY02] also shows that one can at least find the isolated inflection points and vertical tangents of a trinomial curve about as quickly.

Let us conclude our introduction with a recent number-theoretic parallel: It has just been shown by the middle author [Roj02] that the number of **geometrically** isolated[3] roots in $\mathcal{L}^n$ of any μ-sparse $k\times n$ polynomial system, **over any $\mathfrak{p}$-adic field $\mathcal{L}$**, is no more than $1+\left(\mathcal{C}_{\mathcal{L}}n(\mu-n)^3\log\mu\right)^n$, where $\mathcal{C}_{\mathcal{L}}$ is a constant depending only on $\mathcal{L}$ (see also [Roj01] and the references therein for earlier results in this direction). In particular, since $\mathbb{Q}\subset\mathbb{Q}_2$, one thus obtains a bound on the number of isolated roots in $\mathcal{L}^n$ which is polynomial in μ for fixed n, with $\mathcal{L}$ now any fixed number field. One should note that $\mathfrak{p}$-adic fields, just like $\mathbb{R}$, are complete with respect to a suitable metric. So there appears to be a deeper property of metrically complete fields lurking in these quantitative results.

Remark 2. *Domenico Napoletani has recently shown that to calculate $\mathcal{N}'(m_1,\ldots,m_n)$ for any given $(m_1,\ldots,m_n)$, it suffices to restrict to the case of integral exponents [Nap01]. Here, we will bound $\mathcal{N}(n+1,\ldots,n+1,m)$ directly, in the aforementioned cases, without using this reduction.* $\diamond$

1.3 Organization of the Proofs and Obstructions to Extensions

Section 2 provides some background and unites some simple cases where Kushnirenko's conjectured bound in fact holds, and the equalities $\mathcal{K}'(n,\mu)=\mathcal{K}(n,\mu)$ and $\mathcal{N}'(m_1,\ldots,m_n)=\mathcal{N}(m_1,\ldots,m_n)$ are true. We then prove Theorem 1 in Sections 3 and 4, and prove Theorem 2 in Section 5. Proving the (restricted) upper bound on $\mathcal{N}(n+1,\ldots,n+1,m)$ turns out to be surprisingly elementary, but lowering the upper bound on $\mathcal{N}(3,3)$ to 5 then becomes a more involved case by case analysis. Section 6 then applies a variant of the momentum map from symplectic/toric geometry (see, e.g., [Sma70, Sou70] and [Ful93, Sec. 4.2]) to prove Theorem 3.

Section 4 gives an alternative geometric proof that $\mathcal{N}(3,3)\leq 6$. We include this second proof for motivational purposes since it appears to be the first known improvement over $\mathcal{N}'(3,3)\leq 248832$, and since it is the only approach we know which yields part (c) of Theorem 1.

The reader should at this point be aware that our results can of course be combined and interweaved to generate much more complicated examples (with more monomial terms, more complicated supports, and more variables) which admit upper bounds on the number of roots in $\mathbb{R}_+^n$ significantly

[3] A root is geometrically isolated iff it is a zero-dimensional component of the underlying zero set in $\bar{\mathcal{L}}^n$, where $\bar{\mathcal{L}}$ is the algebraic closure of $\mathcal{L}$.

sharper than Khovanski's Theorem on Real Fewnomials (see, e.g., Theorem 4 and the paragraph after in the next section). Nevertheless, it should also be clear that there are still many simple fewnomial systems where nothing better than Khovanski's bound is available, e.g., the exact values of $\mathcal{N}'(4,4)$ and $\mathcal{N}(4,4)$ remain unknown. So let us close with some brief remarks on the obstructions to extending Theorem 1 to more complicated fewnomial systems. In particular, the two main techniques we use are (A) a recursion involving derivatives of certain analytic functions, and (B) an extension of Rolle's Theorem (cf. Section 2) to intersections of lines with certain fewnomial curves.

Our technique from (A) succeeds precisely because the underlying recursion stops in a number of steps depending only on m and n. In particular, while one can apply the same technique to certain slightly more complicated systems (cf. the proof of part (b) of Theorem 1 in Section 3), applying the same technique to a system of type $(4,m)$ results in a much more complicated recursion which won't terminate without strong restrictions on the exponents; and even then the number of steps begins to depend on the exponents. The geometric reason for this is that we in essence project our roots to a line to start our recursion, and such projected roots appear to satisfy sufficiently simple equations just for the systems defined in part (b) (see Remark 4 of the next section).

Our technique from (B) succeeds for the systems (f_1, f_2) coming from $(\star)$ precisely because (i) $Z_+(f_1)$ is diffeomorphic to a line in a very special way, and (ii) the equations arising from checking inflection points and vertical tangents of $Z_+(f_2)$ have a fewnomial structure very similar to that of f_2. In particular, for the systems in $(\star)$, we construct our stated bound by a special application of Bernstein's Theorem [BKK76] in the 2×2 case. However, increasing the number of variables of p (i.e., the number of monomial terms of f_2) leaves us with a system of equations apparently not reducible to Bernstein's Theorem.

Nevertheless, we suspect that there are many similar improvements to Fewnomial Theory over $\mathbb{R}$ which are quite tractable, and we hope that our paper serves to inspire more activity in this area.

2 The Pyramidal, Simplicial, and Zero Mixed Volume Cases

Let us first note some simple inequalities relating the quantities $\mathcal{K}'$, $\mathcal{K}$, $\mathcal{N}'$, and $\mathcal{N}$.

Proposition 1. *We have* $(\mu-1)^n \leq \mathcal{K}'(n,\mu) \leq \mathcal{K}(n,\mu)$,
$$\mathcal{N}'(m_1,\ldots,m_n) \leq \mathcal{K}'(n, m_1+\cdots+m_n-n+1) \leq \mathcal{N}'(\underbrace{m_1+\cdots+m_n-2n+2,\ldots,m_1+\cdots+m_n-2n+2}_{n}),$$
and
$$\mathcal{N}(m_1,\ldots,m_n) \leq \mathcal{K}(n, m_1+\cdots+m_n-n+1) \leq \mathcal{N}(\underbrace{m_1+\cdots+m_n-2n+2,\ldots,m_1+\cdots+m_n-2n+2}_{n}),$$
where we set $\mathcal{N}(m_1,\ldots,m_n) = \mathcal{N}'(m_1,\ldots,m_n) = 0$ *if any* m_i *is negative. In particular, by Theorem 1, we thus have* $\mathcal{K}'(2,4) = \mathcal{K}(2,4) = 5$. ■

Indeed, the last two "left-hand" inequalities follow simply by dividing each f_i by a suitable monomial, while Gaussian elimination on the monomial terms of F easily yields the last two "right-hand" inequalities.

Let us next give a simple geometric characterization of certain fewnomial systems that admit easy root counts.

Definition 5. *Let us call any collection* $L_1 \subsetneq \cdots \subsetneq L_n = \mathbb{R}^n$ *of* n *non-empty subspaces of* $\mathbb{R}^n$ *(so that* $\dim L_i = i$ *for all* i*) a* **complete flag**. *Noting that any polytope in* $\mathbb{R}^n$ *naturally generates a subspace of* $\mathbb{R}^n$ *via the set of linear combinations of all* **differences** *of its vertices, let* $F=(f_1,\ldots,f_n)$ *be an* $n\times n$ *fewnomial system and, for all* i*, let* L_i *be the linear subspace so generated by* $\mathrm{Newt}(f_i)$. *We then say that* F *is* **pyramidal** *iff the Newton polytopes of* F *generate a complete flag. Finally, letting* $A:=[a_{ij}]$ *be any real* $n\times n$ *matrix,* $x:=(x_1,\ldots,x_n)$, $y:=(y_1,\ldots,y_n)$, *and* $y^A:=(y_1^{a_{11}}\cdots y_n^{a_{n1}},\ldots,y_1^{a_{1n}}\cdots y_n^{a_{nn}})$, *we call any change of variables of the form* $x=y^A$ *a* **monomial change of variables**. ⋄

For example, the systems from (1) (cf. Section 1.1) are pyramidal, but systems (2) (cf. Section 1.1), (3), (4), and (5) (cf. Section 1.2) are all non-pyramidal. Note in particular that all binomial systems are pyramidal, but a 2×2 fewnomial system of type $(3,3)$ certainly need not be pyramidal.

Pyramidal systems are a simple generalization of the so-called "triangular" systems popular in Gröbner-basis papers on computer algebra. The latter family of systems simply consists of those F for which the variables can be reordered so that for all i, f_i depends only on $x_1,\ldots,x_i$. Put another way, pyramidal systems are simply the image of a triangular system (with **real** exponents allowed) after multiplying the individual equations by arbitrary monomials and then performing a monomial change of variables.

Recall that an **analytic** subset of a domain $U\subseteq\mathbb{R}^n$ is simply the zero set of an analytic function defined on U. We then have the following fact on monomial changes of variables.

Proposition 2. *If $x:=(x_1,\ldots,x_n)\in\mathbb{R}^n_+$ and A is a real invertible $n\times n$ matrix, then $(x^A)^{A^{-1}}=x$ and the* **monomial map** *defined by $x\mapsto x^A$ is an analytic automorphism of the positive orthant. In particular, such a map preserves smooth points, singular points, and the number of compact and non-compact connected components, of analytic subsets of the positive orthant. Furthermore, this invariance also holds for real m-nomial zero sets in the positive orthant.* ■

The assertion on analytic subsets follows easily from an application of the chain rule from calculus, and noting that such monomial maps are also diffeomorphisms. That the same invariance holds for m-nomial zero sets follows immediately upon observing that the substitution $(x_1,\ldots,x_n)=(e^{z_1},\ldots,e^{z_n})$ maps any n-variate real m-nomial to a real analytic function, and noting that the map defined by $(t_1,\ldots,t_n)\mapsto(e^{t_1},\ldots,e^{t_n})$ is a diffeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n_+$.

Remark 3. *The real zero set of x_1+x_2-1, and the change of variables $(x_1,x_2)=\left(\frac{y_1}{y_2},y_1y_2\right)$, show that the number of isolated* **inflection** *points need not be preserved by such a map: the underlying curve goes from having no isolated inflection points to having one in the positive quadrant.* ⋄

We will later need the following analogous geometric extension of the concept of an over-determined system.

Definition 6. *Given polytopes $P_1,\ldots,P_n\subset\mathbb{R}^n$, we say that they have* **mixed volume zero** *iff for some $d\in\{0,\ldots,n-1\}$ there exists a d-dimensional subspace of $\mathbb{R}^n$ containing translates of P_i for at least $d+1$ distinct i.* ⋄

The **mixed volume**, originally defined by Hermann Minkowski in the late $19^{\underline{\text{th}}}$ century, is a nonnegative function defined for all n-tuples of convex bodies in $\mathbb{R}^n$, and satisfies many natural properties extending the usual n-volume. The reader curious about mixed volumes of polytopes in the context of solving polynomial equations can consult [BZ88, Roj99] (and the references therein) for further discussion. A simple special case of an n-tuple of polytopes with mixed volume zero is the n-tuple of Newton polytopes of an $n\times n$ fewnomial system where, say, the variable x_i does not appear. By multiplying the individual m-nomials by suitable monomials, and applying a suitable monomial change of variables, the following corollary of Proposition 2 is immediate.

Corollary 4. *Suppose F is a fewnomial system, with only finitely many roots in the positive orthant, whose n-tuple of Newton polytopes has mixed volume zero. Then F has no roots in the positive orthant.* ■

Indeed, modulo a suitable monomial change of variables, one need only observe that the existence of a single root in the positive orthant implies the existence of an entire ray of roots (parallel to some coordinate axis) in the positive orthant.

We will also need the following elegant extension of Descartes' Rule to real exponents. It's proof involves a very simple induction using Rolle's Theorem (cf. the next section) and dividing by suitable monomials [Kho91] — tricks we will build upon in the next section.

Definition 7. *For any sequence $(c_1,\ldots,c_m)\in\mathbb{R}^m$, its* **number of sign alternations** *is the number of pairs $\{j,j'\}\in\{1,\ldots,m\}$ such that $j<j'$, $c_jc_{j'}<0$, and $c_i=0$ when $j<i<j'$.* ⋄

Univariate Generalized Descartes' Rule of Signs (UGDRS). *Let $c_1,a_1,\ldots,c_m,a_m$ be any real numbers with $a_1<\cdots<a_m$. Then the number of positive roots of $\sum_{i=1}^m c_ix_1^{a_i}$ is at most the number of sign alternations in the sequence $(c_1,\ldots,c_m)$. In particular, $\mathcal{K}'(1,m)=\mathcal{K}(1,m)=\mathcal{N}'(m)=\mathcal{N}(m)=m-1$.* ■

As a warm-up, we can now prove a stronger version of Kushnirenko's conjecture for certain fundamental families of special cases. In particular, we point out that aside from the domain of Theorem 1, the equalities $\mathcal{K}'(n,\mu)=\mathcal{K}(n,\mu)$ and $\mathcal{N}'(m_1,\ldots,m_n)=\mathcal{N}(m_1,\ldots,m_n)$ appear to be known only for the cases stated in UGDRS and assertions (0), (2), and (4) below.

Theorem 4. *Suppose F is an $n\times n$ fewnomial system of type $(m_1,\ldots,m_n)$ (so $m_1,\ldots,m_n\geq 1$) and consider the following independent conditions:*

(a) *The n-tuple of Newton polytopes of F has mixed volume zero.*

(b) *All the supports of F can be translated into a single set of cardinality $\leq n+1$.*

(c) *F is pyramidal.*

Then, following the notation of Theorem 1, we have:

0. *$\mathcal{N}(m_1,\ldots,m_n)$ is 0, 1, or $\prod_{i=1}^{n}(m_i-1)$ if we respectively restrict to case (a), (b), or (c). Also, in all these cases, $\mathcal{N}'(m_1,\ldots,m_n)=\mathcal{N}(m_1,\ldots,m_n)$.*
1. *If (a), (b), or (c) hold then [F has infinitely many roots $\Longrightarrow$ F has* **no** *isolated roots].*
2. *$\mathcal{N}'(m_1,m_2,\ldots,m_n)=\mathcal{N}(m_1,m_2,\ldots,m_n)=0 \Longleftrightarrow$ some m_i is ≤ 1*
3. *$m_1=2 \Longrightarrow$ [$\mathcal{N}(m_1,m_2,\ldots,m_n)=\mathcal{N}(m_2,\ldots,m_n)$ and $\mathcal{N}'(m_1,m_2,\ldots,m_n)=\mathcal{N}'(m_2,\ldots,m_n)$]. In particular, $\mathcal{N}'(2,\ldots,2)=\mathcal{N}(2,\ldots,2)=1$.*
4. *$\mathcal{K}'(n,\mu)=\mathcal{K}(n,\mu)\leq 1 \Longleftrightarrow \mu\leq n+1$, and equality holds iff $\mu=n+1$.*

One should of course note the obvious fact that $\mathcal{N}$ and $\mathcal{N}'$ are symmetric functions in their arguments. Note also that conditions (a), (b), or (c) need not hold in assertions (2)–(4).

Proof of Theorem 4: First note that the Newton polytopes must all be non-empty. The case (a) portion of assertions (0) and (1) then follows immediately from Corollary 4. Note also that the case (a) portion of assertion (0) implies the "$\Longleftarrow$" direction of assertion (2), since the underlying n-tuple of polytopes clearly has mixed volume zero. The "$\Longrightarrow$" direction of assertion (2) then follows easily from our earlier examples from Section 1. The case (b) portion of assertions (0) and (1) follows easily upon observing that F is a linear system of n equations in n monomial terms, after multiplying the individual equations by suitable monomial terms. We can then finish by Proposition 2.

To prove the case (c) portion of assertions (0) and (1), note that the case $n=1$ follows directly from UGDRS. For $n>1$, we have the following simple proof by induction: Assuming the desired bound holds for all $(n-1)\times(n-1)$ pyramidal systems, consider any $n\times n$ pyramidal system F. Then, via a suitable monomial change of variables, multiplying the individual equations by suitable monomials, and possibly reordering the f_i, we can assume that f_1 depends only on x_1. (Otherwise, F wouldn't be pyramidal.) We thus obtain by UGDRS that f_1 has at most m_1-1 positive roots. By back-substituting these roots into $F':=(f_2,\ldots,f_n)$, we obtain a new $(n'-1)\times(n'-1)$ pyramidal fewnomial system of type $(m'_2,\ldots,m'_{n'})$ with $n'\leq n$ and $m'_2\leq m_2,\ldots,m'_{n'}\leq m_{n'}$. By our induction hypothesis, we obtain that each such specialized F' has at most $\prod_{i=2}^{n'}(m'_i-1)$ isolated roots in the positive orthant, and thus F has at most $\prod_{i=1}^{n}(m_i-1)$ isolated roots in the positive orthant. (We already saw in the introduction that this bound can indeed be attained.)

Our recursive formulae for $\mathcal{N}'(2,m_2,\ldots,m_n)$ and $\mathcal{N}(2,m_2,\ldots,m_n)$ from assertion (3) then follow by applying just the first step of the preceding induction argument, and noting that Proposition 2 tells us that our change of variables preserves non-degenerate roots.

Assertion (4) follows immediately from cases (a) and (b) of assertion (0). ■

One can of course combine and interweave families (a), (b), and (c) to obtain less trivial examples where we have exact formulae for $\mathcal{N}(m_1,\ldots,m_n)$ and $\mathcal{K}(n,\mu)$. More generally, one can certainly combine theorems 1 and 4 to obtain bounds significantly sharper than Khovanski's Theorem on Real Fewnomials, free from Jacobian assumptions, for many additional families of fewnomial systems.

3 Substitutions and Calculus: Proving Theorem 1 Minus Part (c)

Let us preface our first main proof with some useful basic results.

Lemma 1. *Let $F=(f_1,\ldots,f_n)$ be any $n\times n$ fewnomial system of type $(m_1,\ldots,m_n)$ with $m_1=1+\dim\mathrm{Newt}(f_1)$. Then there is another $n\times n$ fewnomial system $G=(g_1,\ldots,g_n)$, also of type $(m_1,\ldots,m_n)$, such that G has the same number of non-degenerate (resp. isolated) roots in $\mathbb{R}^n_+$ as F, $g_1:=1\pm x_1\pm\cdots\pm x_{m_1-1}$ (with the signs in g_1* **not** *all "+") and, for all i, g_i has 1 as one of its monomial terms. In particular, for $m_1=3$, we can assume further that $g_1:=1-x_1-x_2$.*

Proof: By dividing each f_i by a suitable monomial term, we can assume that all the f_i possess the monomial term 1. In particular, we can also assume that the origin $\mathbf{O}$ is a vertex of $\mathrm{Newt}(f_1)$. Note also that the sign condition on g_1 must obviously hold, for otherwise the value of g_1 would be positive on the positive orthant. (The refinement for $m=3$ then follows by picking the monomial term one divides f_1 by more carefully.) So we now need only check that the desired canonical form for g_1 can be attained.

Suppose $f_1:=1+c_1x^{a_1}+\cdots+c_{m_1-1}x^{a_{m_1-1}}$. By assumption, $\mathrm{Newt}(f_1)$ is an m_1-simplex with vertex set $\{\mathbf{O},a_1,\ldots,a_{m_1-1}\}$, so $a_1,\ldots,a_{m_1-1}$ are linearly independent. Now pick any $a_{m_1},\ldots,a_n\in\mathbb{R}^n$ so that $a_1,\ldots,a_n$ are linearly independent. The substitution $x\mapsto x^{A^{-1}}$ (with A the $n\times n$ matrix whose columns are $a_1,\ldots,a_n$) then clearly sends $f_1\mapsto 1+c_1x_1+\cdots+c_{m_1-1}x_{m_1-1}$, and Proposition 2 tells us that this change of variables preserves degenerate and non-degenerate roots in the positive orthant. Then, via the change of variables $(x_1,\ldots,x_{m_1-1})\mapsto(x_1/|c_1|,\ldots,x_{m_1-1}/|c_{m_1-1}|)$, we obtain that g_1 can indeed be chosen as specified. (The latter change of variables preserves degenerate and non-degenerate roots in the positive orthant for even more obvious reasons.) ■

Recall that a polynomial $p\in\mathbb{R}[x_1,\ldots,x_n]$ is **homogeneous of degree *D*** iff $p(ax_1,\ldots,ax_n)=a^Dp(x_1,\ldots,x_n)$ for all $a\in\mathbb{R}$.

Proposition 3. *Suppose $p\in\mathbb{R}[S_1,\ldots,S_n]$ is homogeneous of degree $D\geq 0$. Also let $\alpha_1,u_1,v_1,\ldots,\alpha_n,u_n,v_n\in\mathbb{R}$. Then there is a homogeneous $q\in\mathbb{R}[S_1,\ldots,S_n]$, either identically zero or of degree $D+n-1$, such that $\frac{d}{dt}\left(p(u_1+v_1t,\ldots,u_n+v_nt)\prod_{j=1}^n(u_j+v_jt)^{\alpha_j}\right)=q(u_1+v_1t,\ldots,u_n+v_nt)\prod_{j=1}^n(u_j+v_jt)^{\alpha_j-1}$.*

Proof: By the chain-rule, $\frac{d}{dt}\left(p(u_1+v_1t,\ldots,u_n+v_nt)\prod_{j=1}^n(u_j+v_jt)^{\alpha_j}\right)$ is simply
$\left(\sum_{j=1}^n v_jp_j(u_1+v_1t,\ldots,u_n+v_nt)\right)\left(\prod_{j=1}^n(u_j+v_jt)^{\alpha_j}\right)+p(u_1+v_1t,\ldots,u_n+v_nt)\left(\sum_{i=1}^n\alpha_iv_i\frac{\prod_{j=1}^n(u_j+v_jt)^{\alpha_j}}{u_i+v_it}\right)$
where p_i denotes the partial derivative of p with respect to S_i. Factoring out a multiple of $\prod_{j=1}^n(u_j+v_jt)^{\alpha_j-1}$ from the preceding expression, we then easily obtain that we can in fact take
$q(S_1,\ldots,S_n)=(v_1p_1(S_1,\ldots,S_n)+\cdots+v_np_n(S_1,\ldots,S_n))(S_1\cdots S_n)+p(S_1,\ldots,S_n)\left(\sum_{i=1}^n\alpha_iv_i\frac{S_1\cdots S_n}{S_i}\right)$.
So we are done. ■

Rolle's Theorem. *(1691) Let $g:[a,b]\longrightarrow\mathbb{R}$ be any continuous function with a derivative g' well-defined on (a,b). Then g has r roots in $[a,b]\Longrightarrow g'$ has at least $r-1$ roots in (a,b).* ■

Lemma 2. *Let $m\geq 2$. Then for any real $c_1,u_1,v_1,\ldots,c_m,u_m,v_m$ and $[a_{ij}]$, the function*

$$f(t):=\sum_{i=1}^m c_i\prod_{j=1}^n(u_j+v_jt)^{a_{ij}}$$

has no more than $n+\cdots+n^{m-1}$ roots in the open interval $I:=\{t\in\mathbb{R}_+\mid u_j+v_jt>0 \text{ for all } j\}$.

Furthermore, for any $\alpha_1,\ldots,\alpha_n\in\mathbb{R}$, f has exactly r roots in I implies that there exist $\tilde{c}_1,\ldots,\tilde{c}_m\in\mathbb{R}$ such that

$\tilde{f}(t):=\sum_{i=1}^m\tilde{c}_i\prod_{j=1}^n(u_j+v_jt)^{a_{ij}}$ has at least r roots in I, no root of $\tilde{f}$ in I is degenerate, and no root of $\tilde{f}$ in I is an isolated root of $\left(\left(\prod_{j=1}^m(u_j+v_jt)^{\alpha_j}\right)\tilde{f}'\right)'$.

Proof: Throughout this proof let us consider only those roots lying in the open interval I and assume that f has exactly r roots in I. We will in fact prove a stronger statement involving an extra parameter D and then derive our lemma as the special case $D=0$.

First note that if

$$g(t):=\sum_{i=1}^{m} p_i(u_1+v_1t,\ldots,u_n+v_nt)\prod_{j=1}^{n}(u_j+v_jt)^{a_{ij}}$$

for some homogeneous polynomials $p_1,\ldots,p_m$ of degree D, then

$$g_0(t):=p_1(u_1+v_1t,\ldots,u_n+v_nt)+\sum_{i=2}^{m} p_i(u_1+v_1t,\ldots,u_n+v_nt)\prod_{j=1}^{n}(u_j+v_jt)^{a_{ij}-a_{1j}}$$

has the same number of roots in I as g. In particular, using $D+1$ applications of Rolle's Theorem and Proposition 3, it is clear that $g_1:=g_0^{(D+1)}$ has at least $r-(D+1)$ roots, and we can in fact write

$$g_1(t):=\sum_{i=1}^{m-1} q_i(u_1+v_1t,\ldots,u_n+v_nt)\prod_{j=1}^{n}(u_j+v_jt)^{a'_{ij}},$$

for some array $[a'_{ij}]$ and homogeneous polynomials $q_1,\ldots,q_m$ of degree $D+(D+1)(n-1)$.

Now let $A(m,D)$ denote the maximum number of isolated roots of g in the interval I. By what we've just observed, we immediately obtain the inequality

$$A(m,D)\leq A(m-1,nD+n-1)+D+1,$$

valid for all $m\geq 2$, $n\geq 1$, and $D\geq 0$. That $A(1,D)\leq D$ is clear, so one can then begin to bound $A(m,D)$ for general m by recursion. A simple guess followed by an easy proof by induction yields

$$A(m,D)\leq (1+n+\cdots+n^m)(D+1)-1,$$

which is valid for all $m,n\geq 1$ and $D\geq 0$. So the first assertion is proved.

To prove the second part, note that the first part of our lemma implies that f has only finitely many **critical values** (i.e., values $f(x)$ with $f'(x)=0$) — no more than $n+\cdots+n^{m-1}$, in fact. Similarly, for any $\alpha_1,\ldots,\alpha_m\in\mathbb{R}$, there will only be finitely many roots for $\left(\left(\prod_{j=1}^{m}(u_j+v_jt)^{\alpha_j}\right)f'\right)'$, unless this function is identically zero. In the latter case, no root of $\left(\left(\prod_{j=1}^{m}(u_j+v_jt)^{\alpha_j}\right)f'\right)'$ is isolated. So let us pick $\alpha_1,\ldots,\alpha_m\in\mathbb{R}$ so that $\left(\left(\prod_{j=1}^{m}(u_j+v_jt)^{\alpha_j}\right)f'\right)'$ is not identically zero.

Note then that for all $\delta\in\mathbb{R}^*$ with $|\delta|$ sufficiently small, $f-\delta\prod_{j=1}^{n}(u_j+v_jt)^{a_{1j}}$ will have **no** degenerate roots in I and **no** roots in I making $\left(\left(\prod_{j=1}^{m}(u_j+v_jt)^{\alpha_j}\right)f'\right)'$ vanish. We can in fact guarantee that $f-\delta\prod_{j=1}^{n}(u_j+v_jt)^{a_{1j}}$ will also have at least r non-degenerate roots in I as follows: Let n_+ (resp. n_-) be the number of roots $t\in I$ of f with $f'(t)=0$ and $f''(t)>0$ (resp. $f''(t)<0$). Clearly then, for all $\delta\in\mathbb{R}^*$ with $|\delta|$ sufficiently small, $f-\delta\prod_{j=1}^{n}(u_j+v_jt)^{a_{1j}}$ will have exactly $r+n_--n_+$ or $r+n_+-n_-$ roots in I, according as $\delta>0$ or $\delta<0$. (This follows easily upon dividing through by $\prod_{j=1}^{n}(u_j+v_jt)^{a_{1j}}$.) So let $\tilde{\delta}$ be sufficiently small, and of the correct sign, so that $f-\tilde{\delta}\prod_{j=1}^{n}(u_j+v_jt)^{a_{1j}}$ has at least r roots in I, **no** degenerate roots, and **no** roots making $\left(\left(\prod_{j=1}^{m}(u_j+v_jt)^{\alpha_j}\right)f'\right)'$ vanish.

To conclude, simply let $\tilde{c}_1=c_1-\delta$ and $\tilde{c}_i:=c_i$ for all $i\geq 2$. ■

Proof of Theorem 1 (Minus Part (c)): We will reduce part (a) to part (b), prove part (b), and then refine our argument until we obtain $\mathcal{N}'(3,3)=\mathcal{N}(3,3)=5$.

First note that in part (a), a simple Jacobian calculation reveals that the only way that $Z_+(f_1)$ can be degenerate is if f_1 is the square of a binomial. (Indeed, if $\mathrm{Newt}(f_1)$ is a triangle then Lemma 1 implies that $Z_+(f_1)$ is diffeomorphic to a line.) Part (0) of Theorem 4 then shows that our bound from (a) is easily satisfied in the special case where f_1 is a trinomial with $\mathrm{Newt}(f_1)$ a line segment, so we can assume $\mathrm{Newt}(f_1)$ is a triangle. Since $2+4+\cdots+2^{m-1}=2^m-2$, it then clearly suffices to prove part (b).

To prove part (b), first note that UGDRS implies the case $n=1$, so we can assume $n\geq 2$. Also, from the last paragraph, we already know that we can assume $\mathrm{Vol}(\mathrm{Newt}(f_1))>0$ when $n=2$. Since F has no isolated roots when $n>2$ and the mixed volume of $\mathrm{Newt}(f_1),\ldots,\mathrm{Newt}(f_{n-1})$ is zero (via part (0) of theorem 4 again), we can assume henceforth that the mixed volume of $\mathrm{Newt}(f_1),\ldots,\mathrm{Newt}(f_{n-1})$ is positive. Since the supports of $f_1,\ldots,f_{n-1}$ can then all be translated into the vertex set of an n-simplex, Proposition 2 tells us that we can assume in addition that $f_1,\ldots,f_{n-1}$ are **affine** functions of $x_1,\ldots,x_n$. Letting $f_n(x_1,\ldots,x_n)=\sum_{i=1}^m c_i\prod_{j=1}^n x_j^{a_{ij}}$, we can then simply solve for $x_2,\ldots,x_n$ as functions of x_1 by applying Gaussian elimination to the first $n-1$ equations. Substituting into the last equation we then obtain a bijection between the roots of F in the positive orthant and the roots of $f(t):=\sum_{i=1}^m c_i\prod_{j=1}^n(u_j+v_jt)^{a_{ij}}$ in the interval $I:=\{t\in\mathbb{R}_+ \mid u_j+v_jt>0 \text{ for all } j\}$, where $u_1,v_1,\ldots,u_n,v_n$ are suitable real constants.

A simple Jacobian calculation then yields that $(\zeta_1,\ldots,\zeta_n)$ is a degenerate root of F iff

$$\left[\sum_{\ell=1}^n v_\ell \left.\frac{\partial f}{\partial x_\ell}\right|_{(\zeta_2,\ldots,\zeta_n)=(u_2+v_2\zeta_1,\ldots,u_n+v_n\zeta_1)}=0 \text{ and } f(\zeta_1)=0\right],$$

and the above assertion is clearly true iff $f'(\zeta_1)=f(\zeta_1)=0$. So degenerate (resp. non-degenerate) roots of our univariate reduction correspond bijectively to degenerate (resp. non-degenerate) roots of F. Part (b) then follows immediately from Lemma 2.

To now prove that $\mathcal{N}(3,3)=5$, thanks to Haas' counter-example, it suffices to show that $\mathcal{N}(3,3)<6$. To do this, let us specialize our preceding notation to $(m,n)=(3,2)$, $(c_1,c_2)=(-A,-B)$, $(u_1,v_1,u_2,v_2)=(0,1,1,-1)$, and $(a_{11},a_{12},a_{21},a_{22})=(a,b,c,d)$, for some $a,b,c,d\in\mathbb{R}$ and positive A and B. (Restricting A,B,u_1,v_1,u_2,v_2 as specified can easily be done simply by dividing f_2 by a suitable monomial term, as in the proof of Lemma 1.) In particular, the open interval I becomes $(0,1)$.

By using symmetry we can then clearly reduce to the following cases:

A. $a,b,c>0$ and $d<0$
B. $a,c>0$ and $b,d<0$
C. $a,b>0$ and $c,d<0$
D. $a,b,c,d>0$
E. $a,b,c,d<0$
F. $a>0$ and $b,c,d<0$
G. $a,d>0$, $b,c<0$
H. At least one of the numbers a,b,c,d is zero.

Let $g(t):=\frac{1}{B}t^{1-c}(1-t)^{1-d}f'(t)$. Then Lemma 2 and our earlier substitution trick tells us that it suffices to show that any

$$f(t):=1-At^a(1-t)^b-Bt^c(1-t)^d,$$

with all roots non-degenerate and no root of f an isolated root of g', always has strictly less than 6 roots in the open interval $(0,1)$. So let r be the maximum number of roots in $(0,1)$ of any such f.

Let us now prove $r<6$ in all 8 cases:

A. $a,b,c>0$, $d<0$:

Let $Q(x)=1-Ax^a(1-x)^b$ and $R(x)=Bx^c(1-x)^d$. The roots of f may be regarded as the intersections in the positive quadrant of the parametrized curves $y=Q(x)$ and $y=R(x)$. Since $\lim_{x\to 0^+}Q(x)=1$, $\lim_{x\to 1^-}Q(x)=1$, $\lim_{x\to 0^+}R(x)=0$, and $\lim_{x\to 1^-}R(x)=\infty$, it is easy to see via the Intermediate Value Theorem of calculus that the number of intersections must be odd. (One need only note that $f=Q-R$ and that the signs of f' at the ordered roots of f are nonzero and alternate.) So $r<6$.

B. $a,c>0$, $b,d<0$:
Almost exactly the same argument as case A will work here. The only difference here is that $\lim_{x\to 1^-}Q(x)=-\infty$.

C. $a,b>0$, $c,d<0$:
See Lemma 4 below.

D. $a, b, c, d > 0$:
See Lemma 5 below.

E. $a, b, c, d < 0$.
Multiplying $f(t)$ by $t^{\max\{-a,-c\}}(1-t)^{\max\{-b,-d\}}$, we can immediately reduce to case D.

F. $a > 0$, $b, c, d < 0$:
See Lemma 6 below.

G. $a, d > 0$, $b, c < 0$:
See Lemma 7 below.

H. At least one of the numbers a, b, c, d is zero:
Use Lemma 3 below, noting that our hypotheses here imply that either F or $\hat{F}$ is a quadratic polynomial.

This concludes the proof of Theorem 1, except for part (c), which we will complete in Section 4. ■

Remark 4. *Note that while we can attempt the same substitution trick for more complicated F, the complexity of the resulting recursion (involving derivatives and Rolle's Theorem) increases substantially. For instance, applying our proof in the special case where $n=2$ and $f_1(x,y)=1+x+cx^a-y$ unfortunately results in taking a number of derivatives which depends on a, thus obstructing a bound on the number of roots which is independent of the exponent vectors.* ⋄

We now detail the lemmata cited above.

Lemma 3. *Following the notation of the proof of Theorem 1, recall that $g(t) := \frac{A}{B}t^{a-c}(1-t)^{b-d}(-a(1-t)+bt) - c(1-t) + dt$ and that r is the number of roots of $f(t) := 1 - At^a(1-t)^b - Bt^c(1-t)^d$ in the open interval $(0,1)$, where f has no degenerate roots and no root of f is an isolated root of g'. Also let $F(u) := -a(a-c)(a-c-1)u^3 + (a-c)[2a(b-d+1)+b(a-c+1)]u^2 + (d-b)[a(b-d+1)+2b(a-c+1)]u + b(b-d)(b-d-1)$, and $\hat{F}(u) := -c(c-a)(c-a-1)u^3 + (c-a)[2c(d-b+1)+d(c-a+1)]u^2 + (b-d)[c(d-b+1)+2d(c-a+1)]u + d(d-b)(d-b-1)$. Finally, let N (resp. M) be the maximum number of non-degenerate roots in $(0,1)$ of g (resp. the maximum of the number of positive roots of F and $\hat{F}$), over all $(a,b,c,d) \in \mathbb{R}$ and $(A,B) \in \mathbb{R}^2_+$. Then $r-3 \leq N-2 \leq M \leq 3$.*

Proof: Just as in the proof of Lemma 2, we easily see by Rolle's Theorem and division by suitable monomials in t and $1-t$ that $r-1$ is no more than the number of roots in $(0,1)$ of g. So $r-1 \leq N$. Note also that, in a similar way, $r-1$ is no more than the number of roots of $\hat{g}(t) := \frac{B}{A}t^{c-a}(1-t)^{d-b}g(t)$ in $(0,1)$, and the latter function has the same number of roots (all of which are of course non-degenerate) in $(0,1)$ as g.

To conclude, simply note that for suitable $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, we have that $F\left(\frac{1-t}{t}\right) = t^\alpha(1-t)^\beta g''(t)$, $\hat{F}\left(\frac{1-t}{t}\right) = t^\gamma(1-t)^\delta \hat{g}''(t)$, and both expressions are cubic polynomials in t. So, by our preceding trick again, $N-2 \leq M$, and thus $r-3 \leq M$. That $M \leq 3$ is clear from the fundamental theorem of algebra. ■

Lemma 4. *Following the notation of Lemma 3, let $T(x) := \frac{A}{B}x^{a-c}(1-x)^{b-d}(bx-a(1-x))$, $S(x) := c-(c+d)x$, $\hat{T}(x) := \frac{B}{A}x^{c-a}(1-x)^{d-b}(dx-c(1-x))$, and $\hat{S}(x) := a-(a+b)x$. Then $[a, b > 0$ and $c, d < 0] \Longrightarrow r < 6$.*

Proof: By Lemma 3, we are done if $M < 3$ or $N < 5$. So let us assume $M = 3$ to derive a contradiction. By Descartes' Rule of Signs (see Section 2 for a generalization), the coefficients of $F(u)$ or $\hat{F}(u)$ (ordered by exponent) must have alternating signs. Thus, since $a, a-c, b, b-d > 0$, we have that $a-c-1$ and $b-d-1$ must have the same sign. We then need to discuss two cases:

- $a-c-1 < 0$ and $b-d-1 < 0$:

  This implies $c-a+1 > 0$ and $d-b+1 > 0$. Consequently, the coefficients of u^3 and u^2 in $\hat{F}(u)$ and $F(u)$ are all positive — a contradiction.

- $a-c-1>0$ and $b-d-1>0$:

  The roots of g in $(0,1)$ can be regarded as intersections of $y=T(x)$ and $y=S(x)$, for $0<x<1$. Since $T(\{0,1\})=0$, $-a(1-x)+bx=(a+b)x-a$ is strictly increasing, and $-a<0$, we must have that there is a smallest positive local minimum c_0 of T with $T(c_0)<0$. Thus for x near c_0, $T''(x)>0$. Since $T''(x)<0$ for $0<x\ll 1$, there is $c^*\in(0,c_0)$ such that $T''(c^*)=0$. Let $(x_1,y_1),\ldots,(x_K,y_K)$ be the intersection points of $y=T(x)$ and $y=S(x)$ with $x_1<x_2<\cdots<x_K$. (A simple Jacobian calculation shows that (x_i,y_i) is a degenerate root $\Longleftrightarrow$ x_i is a degenerate root of g. So every (x_i,y_i) is in fact a non-degenerate root.) Then for all $i\in\{1,\ldots,K-1\}$ there is a $c_i\in(x_i,x_{i+1})$ with $T'(c_i)=-(c+d)>0$, and for all $i\in\{1,\ldots,K-2\}$ there is a $d_i\in(c_i,c_{i+1})$ with $T''(d_i)=0$. Note that $c_0<c_1$. Thus $c^*<d_1$ and therefore $T''(x)=0$ has at least $K-1$ solutions. Since T'' and F have the same number of positive roots (observing that $T''(u)/F(u)$ is a monomial in u and $1-u$), we must have $N-1\leq K-1\leq 3$. ■

Lemma 5. *Following the notation of Lemma 4, $a,b,c,d>0 \Longrightarrow r<6$.*

Proof: Again, by Lemma 3, we need only show that $M<3$ or $N<5$. So let us assume $M=3$. Then by Descartes' Rule of Signs, $(a-c)(a-c-1)$ and $(b-d)(b-d-1)$ in the coefficients of u^3 and u^0 in $F(u)$ must have the same sign. There are now four cases to be examined.

- The signs of $a-c$, $a-c-1$, $b-d$, and $b-d-1$ are respectively $+,-,+$, and $-$:

  This makes the signs of coefficients of u^3 and u^2 of $F(u)$ both positive.

- The signs of $a-c$, $a-c-1$, $b-d$, and $b-d-1$ are respectively $-,-,+$, and $+$:

  Since $b-d>0$, we have $d-b<0$ and $d-b-1<0$. This makes the constant term of $\hat{F}(u)$ positive, and hence, the coefficients of u and u^2 of $\hat{F}(u)$ must respectively be negative and positive. That is, $c(d-b+1)+2d(c-a+1)<0$ and $2c(d-b+1)+d(c-a+1)>0$. Thus, $-c(d-b+1)+d(c-a+1)<0$. This is false, since $b-d-1>0$ and $a-c-1<0$.

- The signs of $a-c$, $a-c-1$, $b-d$, and $b-d-1$ are all negative:

  By Descartes' rule of signs, $d-b-1$ and $c-a-1$ in the coefficients of y^3 and y^0 of $\hat{F}(y)$ must have the same sign. If both are negative, then coefficients of u^3 and u^2 of $F(u)$ would both be negative. Thus $d-b-1>0$ and $c-a-1>0$. It is easy to see that $\hat{T}(x)<0$ for $0<x\ll 1$ and $\hat{T}(x)>0$ for $0<1-x\ll 1$ and $\lim_{x\to 0^+}\hat{T}(x)=\lim_{x\to 1^-}\hat{T}(x)=0$. Now let $L_0=\min\{c \mid 1>c>0,\ \hat{T}(c)<0 \text{ and } c \text{ is a local minimum of } \hat{T}\}$ and $U_0=\max\{c \mid 1>c>L_0,\ c \text{ is a local maximum of } \hat{T}\}$. Then for x near L_0, $\hat{T}''(x)>0$. Since $\hat{T}''(x)<0$ for $0<x\ll 1$, there exists $L_1\in(0,L_0)$ such that $\hat{T}''(L_1)=0$. Similarly, there is a $U_1\in(U_0,1)$ such that $\hat{T}''(U_1)=0$.

  The roots of $\frac{B}{A}t^{c-a}(1-t)^{d-b}g$ can be studied via the intersections of $y=\hat{T}(x)$ and $y=\hat{S}(x)$, for $0<x<1$. Let $(x_1,y_1),\ldots,(x_k,y_k)$ be these intersection points, where $x_1<x_2<\cdots<x_k$. (A simple Jacobian calculation shows that (x_i,y_i) is a degenerate root $\Longleftrightarrow$ x_i is a degenerate root of g. So every (x_i,y_i) is in fact a non-degenerate root.) Then there are $c_1,\ldots,c_{k-1}$ with $c_i\in(x_i,x_{i+1})$ and $\hat{T}'(c_i)=-(a+b)<0$ for all $i\in\{1,\ldots,k-1\}$, and $d_1,\ldots,d_{k-2}$ with $d_i\in(c_i,c_{i+1})$ and $\hat{T}''(d_i)=0$ for all $i\in\{1,\ldots,k-2\}$. If $x_1>L_0$, then $L_1<d_1$. If $x_1<L_0$, then $T(x_1)<0$. This implies $T(x_i)<0$ for all $i\in\{1,\ldots,k-2\}$, since the slope $-(a+b)$ of $\hat{S}(x)$ is negative. Therefore, $x_{k-2}<U_0$ and hence $d_{k-2}<U_1$. So $\hat{T}''(x)=0$ has at least $k-1$ solutions. Since $\hat{T}''(x)=0$ and $\hat{F}(y)=0$ have the same number of solutions, we have $N-1\leq k-1\leq M=3$.

- The signs of $a-c$, $a-c-1$, $b-d$, and $b-d-1$ are all positive:

  Since $a-c-1>0$ and $b-d-1>0$, the proof follows almost exactly the same line of reasoning as the last case, by intersecting the graphs of T and S instead of $\hat{T}$ and $\hat{S}$. ■

Lemma 6. *Following the notation of Lemma 4, [$a>0$ and $b,c,d<0$] $\Longrightarrow r<6$.*

Proof: Once again, by Lemma 3, it suffices to show that $M<3$ or $N<5$. So let us assume that $M=3$. By checking the coefficients of u^3 and u^0 in $F(u)$, Descartes' Rule of Signs tells us that $a-c-1$ and $(b-d)(b-d-1)$ must have different signs. There are now three cases to be examined.

- $a-c-1$, $b-d$, and $b-d-1$ are all negative:

  Then the signs of the coefficients of both u^3 and u^2 in $\hat{F}(u)$ will all be positive.

- The signs of $a-c-1$, $b-d$, and $b-d-1$ are respectively $-,+,$ and $+$:

  Multiplying f by $t^{-c}(1-t)^{-d}$ yields $w(t):=t^{-c}(1-t)^{-d}-At^{a-c}(1-t)^{b-d}-B$, where $-c>0$, $a-c>0$, $-d>1$, and $-d+b>1$. The roots of w in $(0,1)$ can be regarded as the intersections of the parametrized curves $y=v(x):=x^{-c}(1-x)^{-d}-Ax^{a-c}(1-x)^{b-d}$ and $y=B$. Let $(x_1,y_1),\ldots,(x_n,y_n)$ be the intersection points of these two curves, where $x_1<x_2<\cdots<x_n$. (A simple Jacobian calculation shows that (x_i,y_i) is a degenerate root $\Longleftrightarrow x_i$ is a degenerate root of f. So every (x_i,y_i) is in fact a non-degenerate root.) Then for all $i\in\{1,\ldots,n-1\}$ there is a $c_i\in(x_i,x_{i+1})$ such that $v'(c_i)=0$. Thus v' has at least $n-1$ roots in $(0,1)$. A straightforward computation then yields,
  $v'(x):=Ax^{a-c-1}(1-x)^{b-d-1}(-(a-c)(1-x)+(b-d)x)+x^{-c-1}(1-x)^{-d-1}(-c(1-x)+dx)$, which clearly has the same number of roots in $(0,1)$ as

  $$t(x):=Ax^a(1-x)^b(-(a-c)(1-x)+(b-d)x)-c(1-x)+dx.$$

  Thus t'' has at least $n-3$ roots in $(0,1)$. Since
  $t''(x)/A=x^{a-2}(1-x)^{b-2}[-(a-c)a(a-1)(1-x)^3+a((a+1)(b-d)+2(b+1)(a-c))x(1-x)^2$
  $-b((b+1)(a-c)+2(b-d)(a+1))x^2(1-x)+(b-d)b(b-1)x^3]$,
  t'' has as many roots in $(0,1)$ as

  $$\begin{aligned}P(u):=\ &-(a-c)a(a-1)u^3+a((a+1)(b-d)+2(b+1)(a-c))u^2\\ &-b((b+1)(a-c)+2(b-d)(a+1))u+(b-d)b(b-1)\end{aligned}$$

  has positive roots. Since $a-1<a-c-1<0$, the coefficients of u^3 and u^0 in $P(u)$ are both positive. Thus P has at most 2 positive roots and we obtain $n-3\leq 2$.

- The signs of $a-c-1$, $b-d$, and $b-d-1$ are respectively $+,+,$ and $-$:

  Since $a-c-1>0$ and $b-d>0$, it is easy to see that $T(x)<0$ for $0<x\ll 1$ and $\lim_{x\to 1^-}T(x)=-\infty$. If $T(x)$ has no local minimum, then $y=T(x)$ and $y=S(x)$ have at most one intersection point. Otherwise, let $c_0:=\min\{c\mid 1>c>0,\ c \text{ is a local minimum of } T\}$. The rest of the proof is similar to that of Lemma 4. ■

Lemma 7. *Following the notation of Lemma 4, [$a,d>0$ and $b,c<0$] $\Longrightarrow r<6$.*

Proof: One last time, Lemma 3 tells us that it suffices to prove that $M<3$ or $N<5$. So let's assume that $M=3$. Checking signs of coefficients of u^3 and u^0 of both $F(u)$ and $\hat{F}(u)$, Descartes' Rule of Signs tells us that $a-c-1<0$ and $d-b-1<0$. On the other hand, the alternating signs of coefficients of u^2 and u^1 of $F(u)$ yield

$$2a(b-d+1)+b(a-c+1)<0 \quad \text{and} \quad a(b-d+1)+2b(a-c+1)>0.$$

Thus, $-a(b-d+1)+b(a-c+1)=a(d-1)+b(1-c)>0$. But this is impossible since $d-1<d-1-b<0$, $1-c>0$, $a>0$, and $b<0$. ■

Remark 5. *When $A=1.12$, $B=0.71$, $a=0.5$, $b=0.02$, $c=-0.05$, and $d=1.8$, there are exactly 5 roots of $1-Ax^a(1-x)^b-Bx^c(1-x)^d$ in $(0,1)$: They are, approximately,*

$$\{0.00396494, 0.02986317, 0.4354707, 0.72522344, 0.99620026\}.$$

In particular, this example is nothing more than the univariate reduction from the proof of Theorem 1 applied to a small perturbation of Haas' counter-example. ⋄

4 A Simple Geometric Approach, a Single Hard Case, and the Proof of Part (c) of Theorem 1

Let us begin with an extension of Rolle's Theorem to smooth curves in the plane.

Lemma 8. *Suppose C is a smooth 1-dimensional submanifold of $\mathbb{R}^2$ with:*

1. *At most I inflection points that are isolated (relative to the locus of inflection points).*
2. *At most N non-compact connected components.*
3. *At most V isolated points of vertical tangency.*

Then the maximum finite number of intersections of any line with C is $I+N+V+1$.

Proof: Let S^1 be the realization of the circle obtained by identifying 0 and π in the closed interval $[0,\pi]$. Consider the natural map $\phi : C \longrightarrow S^1$ obtained by $x \mapsto \theta_x$ where θ_x is the angle in $[0,\pi)$ the normal line of x forms with the x_1-axis. We claim that any $\theta \in S^1$ has at most $I+V+1$ pre-images under ϕ.

To see why, note that by assumption we can express C as the union of no more than $I+V+1$ arcs where (a) any distinct pair of arcs is either disjoint or meets at ≤ 2 end-points, and (b) every end-point is either an isolated point of inflection or vertical tangency of C. (This follows easily by considering the graph whose vertices are the underlying inflection and vertical tangency points, and whose vertex adjacencies are determined by path-connectedness.) Calling these arcs **basic arcs**, it is then clear that the interior of any basic arc is homeomorphic (via ϕ) to a connected subset of $S^1 \setminus \{0\}$. We then easily obtain that any $\theta \in S^1$ has at most $I+V+1$ pre-images under ϕ, since each such pre-image belongs to exactly 1 basic arc.

Recall that a **contact point** of a curve C with a differential system $\frac{\partial \vec{X}}{\partial t} = \vec{G}(t)$ is simply a point at which some solution of $\frac{\partial \vec{X}}{\partial t} = \vec{G}(t)$ has a tangent line in common with C. Now note that any line $L_m := \{(x_1, x_n) \in \mathbb{R}^2 \mid m_1x_1 + m_2x_2 = m_0\}$ normal to C forms an angle of $\mathrm{ArcTan}\left(\frac{m_2}{m_1}\right)$ with the x_1-axis. Thus, the number of contact points C has with the differential system

$$\frac{\partial x_1}{\partial t} = m_2 \ , \ \frac{\partial x_2}{\partial t} = -m_1$$

is at most $I+V+1$. By Rolle's Theorem for Dynamical Systems in the Plane (see, e.g., [Kho91, corollary, pg. 23]), we then obtain that the number of intersections of L_m with C is at most $I+N+V+1$, for any real $(m_0, m_1, m_2) \neq (0,0,0)$. So we are done. ■

Remark 6. *The bound from Lemma 8 is tight in all cases. This is easily revealed by the examples in figure 1 below and their obvious extensions. In particular, one can simply append $N-1$ disjoint lines to extend any example with $N=1$ to $N>1$.* ⋄

We are now ready to give a concise geometrically motivated proof of the nearly optimal bound $\mathcal{N}(3,3) \leq 6$. This "second" proof of $\mathcal{N}(3,3) \leq 6$ was actually the original motivation behind this paper and, via a trivial modification, yields the proof of part (c) of Theorem 1 as well.

Short Geometric Proof of $\boldsymbol{\mathcal{N}(3,3) \leq 6}$: Theorem 4 implies that we can assume that f_1 and f_2 have Newton polygons that are each triangles. Lemma 1 of the last section tells us that we can assume that $f_1 = 1 \pm x_1 \pm x_2$, so we need only check the number of intersections of a line with $Z_+(f_2)$. In particular, since $Z_+(f_2)$ is diffeomorphic to a line (thanks to Proposition 2), Theorem 2 tells us that $Z_+(f_2)$ has no more than 3 inflection points and 1 vertical tangent. By Lemma 8, we are done. ■

Proof of Part (c) of Theorem 1: Via a change of variables almost exactly like that of Lemma 1 from the last section, we can assume that $f_1 = 1 \pm x_1 \pm x_2$. From here, we proceed exactly as in our last proof, noting that here $Z_+(f_2)$ instead has no more than $\mathrm{Area}(\mathrm{Newt}(p))$ isolated vertical tangents, $3\mathrm{Area}(\mathrm{Newt}(p))$ isolated inflection points, and $2D$ non-compact components (thanks to Theorem 2). ■

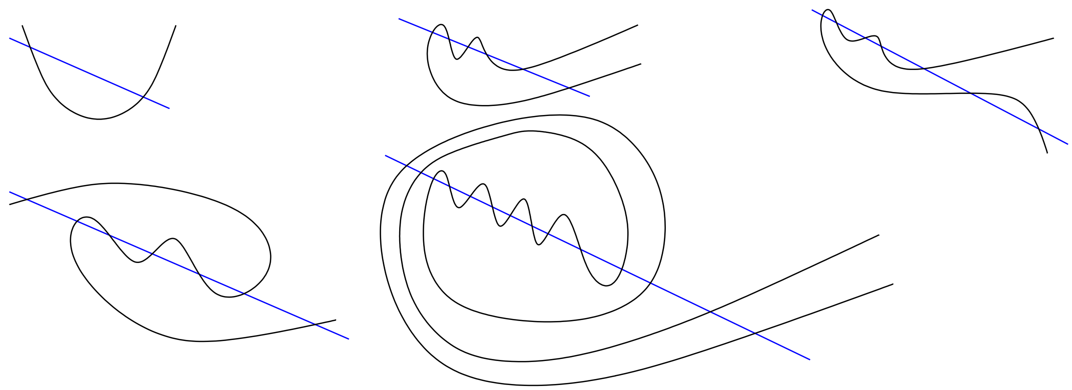

FIGURE 1 Lemma 8 gives a tight bound for $N=1$ and $(I,V)\in\{(0,0),(3,1),(4,1),(3,2),(7,5)\}$; and this generalizes easily to arbitrary (I,N,V).

Fewnomial curves happen to admit a simple "fewnomial" description of their inflection points and singular points. This fact will be used here to prove our classification of when equality holds in our bound $\mathcal{N}(3,3)\leq 5$ (Corollary 1) and in the proof of Theorem 2 in the next section. Let $\partial_i := \frac{\partial}{\partial x_i}$.

Lemma 9. *Suppose $f:\mathbb{R}^2_+ \longrightarrow \mathbb{R}$ is analytic. Then [z is an inflection point or a singular point of $Z_+(f)$] $\Longrightarrow$ $\left\{f(z)=0 \text{ and } [\partial_1^2 f\cdot(\partial_2 f)^2 - 2\partial_1\partial_2 f\cdot\partial_1 f\cdot\partial_2 f + \partial_2^2 f\cdot(\partial_1 f)^2]_{x=z}=0\right\}$. In particular, in the case where $f(x):=p(x^{a_1},\ldots,x^{a_m})$ for some polynomial $p\in\mathbb{R}[S_1,\ldots,S_m]$ and $a_1,\ldots,a_m\in\mathbb{R}^2$, the above cubic polynomial in derivatives is, up to a multiple which is a monomial in (x_1,x_2), a polynomial in $x^{a_1},\ldots,x^{a_m}$ with Newton polytope contained in $3\mathrm{Newt}(p)$.*

Proof: In the case of a singular point, the first assertion is trivial. Assuming $\partial_2 f\neq 0$ at an inflection point then a straightforward computation of $\partial_1^2 x_2$ (via implicit differentation and the chain rule) proves the first assertion. If $\partial_2 f=0$ at an inflection point then we must have $\partial_1 f\neq 0$. So by computing $\partial_2^2 x_1$ instead, we arrive at the remaining case of the first assertion. The second assertion follows routinely from the chain rule. ■

Let us now prove our polygonal classification of bivariate trinomial systems with maximally many roots in the positive quadrant.

Proof of Corollary 1: The segment case follows immediately from Corollary 4. For the remaining cases, Lemma 1 implies that we can assume $f_1:=1\pm x_1\pm x_2$ and $f_2:=1+Ax_1^a x_2^b+Bx_1^c x_2^d$ for some real A and B. In particular, it is easily verified that the number of edges of P_F and P_G are the equal.

So let $S_1:=Ax_1^a x_2^b$, $S_2:=Bx_1^c x_2^d$, and let $Z:=Z_+(f_2)$. Observe that Lemma 9 (along with a suitable rescaling of f_2 and the variables) tells us that we can bound the number of inflection points of Z by analyzing the roots of a homogeneous polynomial in (S_1,S_2) of degree ≤ 3. So let us now explicitly examine this polynomial in our polygonally defined cases.

Clearly then, the triangle case corresponds to setting $a=d>0$ and $b=c=0$. We then obtain that [x is an inflection point or a singular point of Z] $\Longrightarrow$ $1+S_1+S_2=0$ and $S_1+S_2=0$. So Z has **no** inflection points (or singularities). It is also even easier to see that Z has no vertical tangents. So by Lemma 8, $\mathcal{N}(3,3)\leq 2$ in this case. To see that equality can hold in this case, simply consider $F:=(x_1^2+x_2^2-25, x_1+x_2-7)$, which has $P_F=\mathrm{Conv}(\{(0,0),(3,0),(0,3)\})$ and root set $\{(3,4),(4,3)\}$.

For the quadrilateral case, we clearly have that $\mathrm{Newt}(f_1)$ and $\mathrm{Newt}(f_2)$ have exactly two inner edge normal vectors (with length 1) in common. So let v_i be the vertex of $\mathrm{Newt}(f_i)$ incident to both the edges of $\mathrm{Newt}(f_i)$ with these normals. Clearly then, we can assume that our above application of Proposition 2 (which simply involved dividing the f_i by suitable monomial terms and performing an invertible monomial change of variables) gives us $v_1=\mathbf{O}$ as well. So we can assume $b=c=0$ and $a,d>0$. We then get the pair of equations $1+S_1+S_2=0$ and $a(d-1)S_1-d(a-1)S_2=0$, with $a,d\notin\{0,1\}$. (If $\{a,d\}\cap\{0,1\}\neq\emptyset$ then F, or a suitable pair of linear combination of F, would be pyramidal and we would be done by Theorem 4.) So Z can have at most 1 inflection point. It is also even easier to see that Z has no vertical tangents. So by another application of Lemma 8, $\mathcal{N}(3,3)\leq 4$ in this case. To see that equality can hold in this case, simply consider the system $(x_1^2-3x_1+2, x_2^2-3x_2+2)$, which has $P_F=\mathrm{Conv}(\{\mathbf{O},(2,0),(2,2),(0,2)\})$ and root set $\{(1,1),(1,2),(2,1),(2,2)\}$.

As for the pentagonal case, we can again assume (just as in the quadrilateral case) that our application of Proposition 2 placed the correct vertex of $\mathrm{Newt}(f_1)$ at the origin. In particular, we can assume $b=0$ and $a,c,d>0$. We then get the pair of equations $1+S_1+S_2=0$ and $a^2(d-1)S_1^2+a(ad-d-2c)S_1S_2-c(c+d)S_2^2=0$, with $ac(d-1)(c+d)\neq 0$. (Similar to the last case, it is easily checked that if the last condition were violated, then we would be back in one of our earlier solved cases.) However, a simple check of the discriminant of the above quadratic form in (S_1,S_2) shows that there is at most 1 root, counting multiplicities, in any fixed quadrant. So, similar to the last case, we obtain $\mathcal{N}(3,3)\leq 4$ in this case. To see that the equality can hold in this case, simply consider the system $(x_2^2-7x_2+12, -1+x_1x_2-x_1^2)$, which has $P_F=\mathrm{Conv}(\{\mathbf{O},(2,0),(2,2),(1,3),(0,2)\})$ and root set $\left\{\left(3,\frac{3\pm\sqrt{5}}{2}\right),\left(4,2\pm\sqrt{3}\right)\right\}$. ■

5 Monomial Morse Functions and Connected Components: Proving Theorem 2

Let us begin with a refinement of a lemma due to Khovanski.[4] Recall that a ***d*-flat** in $\mathbb{R}^n$ is simply a translate of a d-dimensional subspace of $\mathbb{R}^n$.

Lemma 10. *Let $a_1,\ldots,a_m\in\mathbb{R}^n$, $c_1,\ldots,c_m\in\mathbb{R}$, and let Z be the zero set of $\sum_{i=1}^m c_ie^{a_i\cdot z}$ in $\mathbb{R}^n$, where $\cdot$ denotes the usual Euclidean inner product of vectors in $\mathbb{R}^n$. Then there is an $(n-1)$-flat in $\mathbb{R}^n$ which intersects at least half of the non-compact connected components of Z. Furthermore, the set of unit normal vectors of all such $(n-1)$-flats is non-empty and open in the unit $(n-1)$-sphere.*

For the convenience of the reader, we sketch a proof below, based on an argument of Jean-Jacques Risler from [Ris85, Sec. 2].

Proof: Let us call any function of the form described in the statement of the theorem an **exponential *m*-sum**. Let Z_i be any non-compact component of Z and C_i any connected unbounded curve (defined by a system of exponential m-sums) lying in Z_i. Let p_i be any limit point (as $\|z\|\longrightarrow+\infty$) of the set $\left\{\frac{z}{\|z\|}\;\middle|\; z\in C_i\right\}$. That the set of such limit points is in fact finite follows easily from a slightly more general version of Khovanski's Theorem on Real Fewnomials [Kho91, Cor. 6, Pg. 80, Sec. 3.12], stated in terms of exponential sums.

If H is any hyperplane (so $\mathbf{O}\in H$) such that $p_i\not\in H$ for all i then one of the open unit hemispheres defined by H contains at least half of the points p_i. In particular, note that such an H must clearly exist and any hyperplane H' with unit normal vector sufficiently near that of H will also define an open unit hemisphere containing at least half the p_i. To conclude, note that any $(n-1)$-flat, parallel to H and far enough in the direction of the pole of the hemisphere containg the most p_i, will intersect half of the C_i and thus half of the Z_i. So we are done. ■

We will also need an extension of the classical bounds on the number of connected components of a real algebraic set.

Lemma 11. *Given any μ-sparse $k\times n$ fewnomial system F, the number of connected components of $Z_+(F)$ is no more than $2^{n-\frac{1}{2}}(2n+1)^\mu 2^{\mu(\mu+1)/2}$.* ■

The smooth case, which admits a sharper bound, is detailed in [Kho91, Sec. 3.14]. The special case of integral exponents (allowing degeneracy) is nothing more than [Roj00a, Cor. 3.2] and the proof in [Roj00a, Sec. 3.2] extends with no difficulty to real exponents. One can in fact generalize the above lemma to semi-Pfaffian sets, provided one loosens the stated upper bound somewhat [Zel99].

[4] In [Ris85], Risler outlines a proof of the first portion of Lemma 10, in the special case where all the a_i lie in $\mathbb{Z}^n$. He also cites a paper of Khovanski for further details. However, as far as the authors can tell, the wrong paper by Khovanski was cited.

A construction which will prove quite useful when we count components via critical points of maps is to find a monomial map which is a **Morse function** [Mil63] relative to a given fewnomial zero set. Recall that an ***n*-dimensional polyhedral cone** in $\mathbb{R}^n$ is simply a set of the form $\{r_1a_1+\cdots+r_na_n \mid r_1,\ldots,r_n\geq 0\}$, where $\{a_1,\ldots,a_n\}$ is a generating set for $\mathbb{R}^n$. In particular, an n-dimensional cone in $\mathbb{R}^n$ always has non-empty interior.

Lemma 12. *Suppose f is an n-variate m-nomial. Then there is an n-dimensional polyhedral cone $K\subseteq\mathbb{R}^n$ such that $a\in K\setminus\{\mathbf{O}\}$ implies*

1. *Every critical point of the restriction of x^a to $Z_+(f)$ is non-degenerate.*
2. *The level set in $Z_+(f)$ of any regular value of x^a has dimension $\leq n-2$.*
3. *No connected components of $Z_+(f)$ other than isolated points are contained in any level set of x^a.*
4. *At least half of the non-compact connected component of $Z_+(f)$ have unbounded values of x^a.*

Proof: Set $Z:=Z_+(f)$. Let us first show how assertion (3) can be attained: Since the number of components of Z is finite by Lemma 11, we can temporarily assume that Z consists of a single component. Then, if we could find n linearly independent $a\in\mathbb{R}^n$ with $Z\subset\{x\in\mathbb{R}^n_+ \mid x^a=c_a\}$ for some c_a, Proposition 2 would immediately imply that Z is contained in a point. So condition (3) can be enforced.

To ensure the truth of condition (4), note that we can perform the substitution $(x_1,\ldots,x_n)=(e^{z_1},\ldots,e^{z_n})$ to reduce to finding an $(n-1)$-flat intersecting at least half of the non-compact components of the real zero set of an exponential m-sum. That such a hyperplane exists (and the fact that a small open neighborhood of such hyperplanes exist) then follows from Lemma 10, so condition (4) holds.

To enforce conditions (1) and (2) let us maintain our last change of variables. A simple derivative computation (noting that $x\mapsto(e^{x_1},\ldots,e^{x_n})$ defines a diffeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n_+$) then shows that it suffices to instead prove the analogous statement where f is replaced by a real exponential sum (a real analytic function in any event) and x^a is replaced by the linear form $a_1z_1+\cdots+a_nz_n$. The latter analogue is then nothing more than an application of [BCSS98, Lemma 1, Pg. 304]. Since the number of components of Z is finite, we thus obtain assertions (1) and (2).

Our lemma then follows by intersecting the four sets of a we have just determined, and noting (thanks to Lemma 10) that the intersection clearly contains a cone over a small $(n-1)$-simplex. ■
Note that if we omit condition (4) then we can make conditions (1)–(3) hold in an even larger set of a: all a in $\mathbb{R}^n$ outside a finite union of hyperplanes.

Example 3. *In general, one can not find an a with* **every** *component of $Z_+(f)$ giving unbounded values for x^a. This follows from an elementary calculation with $n=2$ and*

$$f(x,y):=\left(1-x-xy-\frac{1}{y}\right)\left(1-y-xy-\frac{1}{x}\right)\left(1-\frac{1}{x}-\frac{1}{y}\right),$$

showing that, for all (a_1,a_2), $x^{a_1}y^{a_2}$ has unbounded values on no more than 2 of the 3 components of $Z_+(f)$. The authors thank Daniel Perrucci and Fernando Lopez Garcia for this example. ◇

We will also need the following useful perturbation result, which can be derived via Sard's Theorem [Hir94] and a simple homotopy argument. (See, e.g., [Bas99, Lemma 2] for even stronger results of this form in the setting of integral exponents and zero sets in $\mathbb{R}^n$.)

Lemma 13. *Let f be any n-variate m-nomial, $Z^\delta_+(f)$ the solution set of $|f|\leq\delta$ in $\mathbb{R}^n_+$, and $\overset{\circ}{Z^\delta_+}(f)$ the boundary of $Z^\delta_+(f)$. Then for $\delta>0$ sufficiently small, $\overset{\circ}{Z^\delta_+}(f)$ is smooth and has at least as many compact (resp. non-compact) connected components as $Z_+(f)$.* ■

Example 4. *If $f(x,y) := x^2 + (1-xy)^2$ then note that $Z_+(f)$ is empty while, for any $\delta > 0$, $Z^\delta_+(f)$ contains the point $\left(\sqrt{\delta}, \frac{1}{\sqrt{\delta}}\right)$. So $Z^\delta_+(f)$ need not have the exactly same number of compact (or non-compact) components as $Z_+(f)$, even if $\delta > 0$ is very small. The authors thank Daniel Perrucci for this example.* $\diamond$

Example 5. *Boundaries of tubes about analytic sets behave a bit differently in $\mathbb{R}^n_+$ than in $\mathbb{R}^n$. For instance, unlike the analogous bound over $\mathbb{R}^n$ (see, e.g., [Roj00a, Lemma 3.1]), the number of components of $Z_+(f)$ can* **not** *be bounded above by* **half** *the number of components of $\overset{\circ}{Z^\delta_+}(f)$: taking $f(x,y) = \prod_{i=1}^D (y - ix)$, it is easily checked that $Z_+(f)$ has exactly D components, while $\overset{\circ}{Z^\delta_+}(f)$ has exactly $D+1$ components for $\delta > 0$ sufficiently small.* $\diamond$

Proof of Theorem 2: For simplicity, let us assume that all components are connected and lie in $\mathbb{R}^n_+$.

Assertion (0) follows immediately from Proposition 2, UGDRS, and noting that 0 is an n-variate polynomial with exactly one non-compact connected component in its real zero set.

To prove assertion (1), note that assertion (2) of Theorem 2 (which we'll soon prove below) and assertion (4) of Theorem 4 easily imply the first formula. Proposition 2 then tells us that the second equality can be proved simply by employing a monomial change of variables to reduce to the case of an $(m-1)$-variate m-nomial g. In particular, since $m \geq 3$, every component of $Z_+(g)$ will still be non-compact. (This is clear from another application of Proposition 2, separating the cases where $\dim \mathrm{Newt}(g)$ equals, or is strictly less than, m.) Moreover, $\dim \mathrm{Newt}(g) = m \Longrightarrow$ the number of non-compact components is exactly 1. So we can assume that $\dim \mathrm{Newt}(g) < m$, use Proposition 2 one last time, and then intersect with an appropriate coordinate flat to derive the final inequality. So assertion (1) is proved and we can assume henceforth that $n \geq 2$.

To prove assertion (2), let us first construct a concrete family of examples realizing the lower bound: Consider the polynomials

$$g_1(x) := \left(\sum_{i=2}^n (x_i - 1)^2\right) + \left(\prod_{i=1}^{\lfloor m/2 \rfloor - n - 1} (x_1 - i)^2\right) \quad \text{and} \quad g_2(x) := \sum_{j=1}^n \prod_{i=1}^{\lfloor (m-1)/(2n) \rfloor} (x_j - i)^2.$$

Clearly, g_1 and g_2 respectively have exactly $2\lfloor m/2 \rfloor$ and $2n\lfloor (m-1)/(2n) \rfloor$ monomial terms. From the basic fact that $a^2 + b^2 = 0 \Longrightarrow a = b = 0$ for all real (a,b), it is easily checked that the numbers of roots of g_1 and g_2 in $\mathbb{R}^n_+$ are finite and in fact are identically the formulae embedded in our lower bound. More precisely, we immediately obtain our lower bound **for a restricted class of m depending on the congruence class of m mod 2 or mod $2n$.** This restriction can easily be removed by adding additional monomial terms in such a way that the number of compact and non-compact components is not decreased. To do this, simply note that by Sard's Theorem [Hir94] (and the definition of an n-sphere), we have that $Z_+(g_i - \delta_0)$ is smooth and has the same numbers of compact and non-compact components as $Z_+(g_i)$ for all $\delta_0 > 0$ sufficiently small. Similarly, the same will then be true of $Z_+(x_1(g_i - \delta_0) - \delta_1)$, for $\delta_1 > 0$ sufficiently small, and the latter polynomial has exactly 1 more monomial term than g_i. Proceeding inductively, we can thus remove our restriction on m, and we thus obtain the lower bound of assertion (2).

To prove the upper bound of assertion (2), note that we can divide by a suitable monomial so that f has a nonzero constant term. By Lemma 13, we then have that for $\delta > 0$ sufficiently small, it suffices to bound the number of compact components of $\overset{\circ}{Z^\delta_+}(f)$ — an "envelope" of $Z_+(f)$. Recall that Lemma 13 also grants us that $\overset{\circ}{Z^\delta_+}$ can be assumed to be smooth.

By Proposition 2 and Lemma 12, we can then pick an $n \times n$ matrix A so that, after we make the change of variables $x = y^A$, the number of compact and non-compact components of $\overset{\circ}{Z^\delta_+}$ is preserved, **no** component of $\overset{\circ}{Z^\delta_+}$ of positive dimension is contained in a hyper-plane parallel to the y_1-coordinate hyperplane, and we can use critical points of the function y_1 to count compact components.

Consider then the systems of equations $G_\pm := (f \pm \delta, y_2\partial_2 f, \ldots, y_n\partial_n f)$, where $\partial_i = \frac{\partial}{\partial y_i}$ here. By construction, every compact component of $\overset{\circ}{Z}{}^{\delta}_{+}(f)$ results in at least two extrema of the function y_1, i.e., $P_{\text{comp}}(n,m)$ is bounded above by an integer no greater than half the total number of roots of G_+ and G_-. (In particular, if $Z_+(f)$ were smooth to begin with, then we could have omitted the use of $\overset{\circ}{Z}{}^{\delta}_{+}(f)$ and $G_\pm$, since $P_{\text{comp}}(n,m)$ would instead be bounded above by an integer no greater than half the number isolated roots of $G := (f, y_2\partial_2 f, \ldots, y_n\partial_n f)$.) Note also that by assertion (1) of Lemma 12, all the roots of $G_\pm$ (and G) are non-degenerate. Furthermore, $G_\pm$ (and G) clearly has no more than m distinct exponent vectors, so the upper bound on $P_{\text{comp}}(n,m)$ holds. As for the number of compact components of $Z_+(\rho)$, the preceding argument applies as well, so we need only observe that $\rho \pm \delta$ and $y_2\partial_2\rho$ are both of the form $q(x^{r_1}y^{s_1}, x^{u_2}y^{v_2})$ where $\text{Newt}(q) = \text{Newt}(p)$. So assertion (2) follows.

To prove assertion (3), let us construct another family of explicit examples: Consider the polynomials

$$h_1(x) := \prod_{i=1}^{m-1}(x_1 - i) \qquad \text{and} \qquad h_2(x) := \sum_{j=1}^{n-1} \prod_{i=1}^{\lfloor (m-1)/(2n-2) \rfloor} (x_j - i)^2.$$

Clearly, h_1 and h_2 respectively have exactly m and $2(n-1)\lfloor (m-1)/(2n-2) \rfloor$ monomial terms. Note also that $Z_+(h_1)$ and $Z_+(h_2)$ have only non-compact connected components, and the numbers of such components are in fact the formulae embedded in our lower bound (for a restricted class of m). The lower bound of assertion (3) then follow easily, mimicking the argument we used earlier to remove the congruence class restriction which arose during the proof of the lower bound of assertion (2).

To prove the upper bounds of assertion (3), let us work directly with f and make an independent application of Lemma 12. We can then apply Proposition 2 to make a change of variables $x = y^{A'}$ (preserving the number of compact and non-compact components of $Z_+(f)$), so that at least half of the non-compact components of $Z_+(f)$ have unbounded values of y_1. So, for $\varepsilon > 0$ sufficiently small, the number of non-compact components of $Z_+(f)$ is no more than twice the number of components of the intersection $Z' := Z_+(f) \cap \{ y \in \mathbb{R}^n_+ \mid y_1 = \frac{1}{\varepsilon} \}$. So by substituting $y_1 = \frac{1}{\varepsilon}$ into f, we obtain a new m-nomial hypersurface $Z'' \subseteq \mathbb{R}^{n-1}$ with at least as many components as Z'. So Z'' has at least half as many components as $Z_+(f)$ has non-compact components, and thus the number of non-compact components of $Z_+(f)$ is no more than $2P(n-1,m)$. So the upper bound on $P_{\text{non}}(n,m)$ is proved. As for the number of non-compact connected components of $Z_+(\rho)$, the preceding argument still applies. So we need only observe that, modulo a monomial change of variables via Proposition 2, ρ can be assumed to be a polynomial of degree D. Lemma 10 and Bézout's Theorem [Sha77, ex. 1, pg. 198] (along with an exponential change of variables) then proves what is left of assertion (3).

To prove assertion (4) simply note that the isolated points of vertical tangency of $Z_+(f)$ are exactly the isolated roots of the bivariate fewnomial system $H := \left(f, x_2\frac{\partial f}{\partial x_2}\right)$. When $f(x) = p(x^{a_1}, \ldots, x^{a_m})$ for some $p \in \mathbb{R}[S_1, \ldots, S_m]$, a simple application of the chain rule then shows that $x_2\frac{\partial f}{\partial x_2} = q(x^{a_1}, \ldots, x^{a_m})$ for some $a_1, \ldots, a_m \in \mathbb{R}^2$ and some $q \in \mathbb{R}[S_1, \ldots, S_m]$ with $\text{Newt}(q) \subseteq \text{Newt}(p)$. In particular, $p = 1 + S_1 + \cdots + S_m \Longrightarrow q$ is a homogeneous linear form in $S_1, \ldots, S_m$, so the first part of assertion (4) follows. To prove the second part, note that (r_1, s_1) and (u_2, v_2) linearly dependent $\Longrightarrow Z_+(\rho)$ is a union of no more than two binomial curves (via Proposition 2 and factoring over $\mathbb{R}$), and such curves have no isolated points of vertical tangency in $\mathbb{R}^2_+$. So, assuming $\det\begin{bmatrix} r_1 & u_2 \\ s_1 & v_2 \end{bmatrix} \neq 0$, Proposition 2 then tells us that it suffices to count the isolated roots $(S_1, S_2) \in \mathbb{R}^2_+$ of the 2×2 polynomial system H. By Bernstein's Theorem [BKK76], the number of complex isolated roots of the resulting system is at most Area(Newt(p)). So assertion (4) is proved.

To prove assertion (5), first note that $[m = 1 \Longrightarrow Z_+(f)$ is empty$]$ and $[m = 2 \Longrightarrow Z_+(f)$ has no isolated inflection points$]$. So we can assume $m = 3$ and that f has a constant term. Note then that by Lemmata 1 and 9, (x_1, x_2) is an inflection point of $Z_+(f) \Longrightarrow f(x) = q(x^{r_1}y^{s_1}, x^{u_2}y^{v_2}) = 0$, where $q \in \mathbb{R}[S_1, S_2]$. In particular, Lemma 9 tells us that q is either a homogeneous cubic polynomial or a polynomial with Newton polytope contained in $3\text{Newt}(p)$, according as we focus on the first or second part of assertion (5). Just as in the last paragraph, we can also assume that $\det\begin{bmatrix} r_1 & u_2 \\ s_1 & v_2 \end{bmatrix} \neq 0$, and thus reduce to counting the isolated roots in $\mathbb{R}^2_+$ of a 2×2 polynomial system in (S_1, S_2). For the first part of assertion (5), the fundamental theorem of algebra tells us that q splits completely over $\mathbb{C}[S_1, S_2]$, so we can further reduce to no more than three 2×2 linear systems and easily obtain our bound of $3\mathcal{K}'(2,m)$. For the second part, we can easily conclude by Bernstein's Theorem [BKK76]. So assertion (5) is proved.

To prove the last observation of Theorem 2, note that by Proposition 1 and Lemma 1, $Z_+(f)$ has a singularity $\Longrightarrow$ $\mathrm{Newt}(f)$ is a line segment, and then f must be the square of a binomial. So the case where $Z_+(f)$ is singular follows immediately. The case where $Z_+(f)$ is smooth then follows easily from assertions (4) and (5), since Theorem 4 of Section 2 implies that $\mathcal{K}'(2,3) = \mathcal{K}(2,3) = 1$. ■

6 Momenta, Polytopes, and the Proof of Theorem 3

Let $\bar{S}$ and $\mathrm{Int}(S)$ respectively denote the topological closure and topological interior of any set S, and let $\mathrm{RelInt}(Q)$ denote the relative interior of any d-dimensional polytope $Q\subset\mathbb{R}^n$, i.e., $Q\setminus R$ where R is the union of all faces of Q of dimension strictly less than d (using $\emptyset$ as the only face of dimension <0). We then have the following variant of the momentum map from symplectic geometry [Sma70, Sou70].

Lemma 14. *Given any n-dimensional convex compact polytope $P\subset\mathbb{R}^n$, there is a real analytic diffeomorphism $\psi_P : \mathbb{R}^n_+ \longrightarrow \mathrm{Int}(P)$. In particular, if f is an n-variate m-nomial with $\mathrm{Newt}(f) = P$ (so $\dim \mathrm{Newt}(f) = n$) and $w \in \mathbb{R}^n \setminus \{\mathbf{O}\}$, then $\psi_P(Z_+(f))$ has a limit point in $\mathrm{RelInt}(P^w) \Longrightarrow \mathrm{Init}_w(f)$ has a root in $\mathbb{R}^n_+$. Moreover, there is a real analytic diffeomorphism between $Z_+(\mathrm{Init}_w(f)) \subset \mathbb{R}^n_+$ and $\left(\mathrm{RelInt}(P^w)\cap\overline{\psi_P(Z_+(f))}\right)\times\mathbb{R}^{n-\dim P^w}_+$.*

Proof: By [Ful93, Sec. 4.2, Lemma, Pg. 82], the map $\phi : \mathbb{R}^n \longrightarrow \mathrm{Int}(P)$ defined by

$$\phi(x) := \sum_{p \text{ a vertex of } P} p e^{p\cdot x} \Bigg/ \sum_{q \text{ a vertex of } P} e^{q\cdot x}$$

is a real analytic diffeomorphism. Composing coordinate-wise with the logarithm function, we then obtain that

$$\psi_P(x) := \sum_{p \text{ a vertex of } P} p x^p \Bigg/ \sum_{q \text{ a vertex of } P} x^q$$

yields our desired real analytic diffeomorphism from $\mathbb{R}^n_+$ to $\mathrm{Int}(P)$.

The remainder of the lemma follows easily via a monomial change of variables. In particular, the special case where P can be defined by a finite set of inequalities with rational coefficients is already embedded in the theory of toric varieties, e.g., [Ful93, Prop., Pg. 81]. The general case of arbitrary polytopes in $\mathbb{R}^n$ can be proved as follows: Let $d := \dim P^w$, let $v_1, \ldots, v_{n-d} \in \mathbb{Q}^n$ be any linearly independent normal vectors of P^w, and let $v_{n-d+1}, \ldots, v_n \in \mathbb{Q}^n$ be any linearly independent vectors parallel to P^w. Then, letting A be the inverse of the $n\times n$ matrix whose $i^{\underline{\text{th}}}$ column is v_i for all $i\in\{1,\ldots,n\}$, we can clearly write $f(y^A) = g(y_1,\ldots,y_n)$, where

$$g(y_1,\ldots,y_n) := \sum_{\alpha := (\alpha_1,\ldots,\alpha_{n-d})} y_1^{\alpha_1}\cdots y_{n-d}^{\alpha_{n-d}} g_\alpha(y_{n-d+1},\ldots,y_n),$$

the sum ranges over $\{(v_1\cdot a,\ldots,v_{n-d}\cdot a) \mid a\in\mathrm{Supp}(f)\}$, and for any such α there is an m_α such that g_α is a d-variate m_α-nomial. Most importantly, if

$$\beta = \left(\min_{a\in\mathrm{Supp}(f)}\{v_1\cdot a\},\ldots,\min_{a\in\mathrm{Supp}(f)}\{v_{n-d}\cdot a\}\right)$$

then $(y_1,\ldots,y_{n-d})^\beta g_\beta(y_{n-d+1},\ldots,y_n)=\mathrm{Init}_w(f)(y^A)$. Clearly then, $\psi_P(Z_+(f))$ has a limit point in $\mathrm{RelInt}(P^w) \Longrightarrow$ there is an $M>0$ such that $Z_+(f)$ intersects

$$\left(\bigcap_{i=n-d+1}^{n}\left\{x\in\mathbb{R}^n_+ \mid x^{v_i}>\frac{1}{M}\right\}\right)\cap\left(\bigcap_{i=n-d+1}^{n}\left\{x\in\mathbb{R}^n_+ \mid x^{v_i}<M\right\}\right)\cap\bigcap_{i=1}^{n-d}\left\{x\in\mathbb{R}^n_+ \mid x^{v_i}=\varepsilon_i\right\}$$

for all $\varepsilon_i>0$ sufficiently small, since P^w is compact and ψ_P is a diffeomorphism. By Proposition 2, the map $x\mapsto x^A$ is a diffeomorphism, so there is also an $M'>0$ such that $Z_+(g)$ intersects

$$\left(\bigcap_{i=n-d+1}^{n}\left\{y\in\mathbb{R}^n_+ \mid y_i>\frac{1}{M'}\right\}\right)\cap\left(\bigcap_{i=n-d+1}^{n}\left\{y\in\mathbb{R}^n_+ \mid y_i<M'\right\}\right)\cap\left\{y_i=\delta_i \mid i\in\{1,\ldots,n-d\}\right\}$$

for all $\delta_i>0$ sufficiently small. By our formula relating g and g_β (and the fact that $Z_+(g)$ is locally closed, being the zero set of a continuous function), we then have that $Z_+(g_\beta)$ intersects the last set as well. So $Z_+(g_\beta)$, and thus $Z_+(\mathrm{Init}_w(f))$, is non-empty.

To conclude, a routine monomial change of variables shows that

$$\psi_w(x):=\left(\sum_{p \text{ a vertex of } P^w} p x^p \Bigg/ \sum_{q \text{ a vertex of } P^w} x^q\right)\times(x^{v_1},\ldots,x^{v_{n-d}})$$

gives us our desired real analytic diffeomorphism. ■
Note that the converse of Lemma 14 need not hold: A simple counter-example is

$$f(x,y)=(x^2+y^2-1)^2+(x-1)^2 \text{ and } w=(0,1).$$

We also point out that the easiest way to understand the above lemma is to take any example f with Newton polytope identical (near the origin) to the nonnegative orthant, and then note that one is in essence compactifying $\mathbb{R}^n_+$ by adding coordinate subspaces, as well as some other pieces which are images of $(\mathbb{R}^*)^k$ under monomial maps. Indeed, the monomial change of variables in our proof essentially results in an invertible affine map which sends a d-dimensional face of P to a d-dimensional coordinate subspace of $\mathbb{R}^n$.

Theorem 3 then follows easily from a refinement of the last lemma.

Lemma 15. *Following the notation of Lemma 14, assume that $Z_+(\mathrm{Init}_w(f))$ is smooth for all $w\in\mathbb{R}^n\setminus\{\mathbf{O}\}$. Then*

1. *For any facet Q of P, every connected component of $\mathrm{RelInt}(Q)\cap\overline{\psi_P(Z_+(f))}$ is an $(n-2)$-manifold which is the set of limit points in $\mathrm{RelInt}(Q)$ of $\psi_P(C)$ for some* **unique** *non-compact connected component C of $Z_+(f)$.*

2. *C a non-compact connected component of $Z_+(f)\Longrightarrow\psi_P(C)$ has a limit point in $\mathrm{RelInt}(Q)$ for some inner* **facet** *Q of P.*

Proof: To prove (1), first note that the last portion of Lemma 14 already tells us that every connected component of $\mathrm{RelInt}(Q)\cap\overline{\psi_P(Z_+(f))}$ is an $(n-2)$-manifold, since $Z_+(\mathrm{Init}_w(f))$ is smooth for all w. (Indeed, the number of connected components of any $Z_+(\mathrm{Init}_w(f))$, and thus $\mathrm{RelInt}(Q)\cap\overline{\psi_P(Z_+(f))}$, is finite by Lemma 11.) Furthermore, it is clear that every connected component of $\mathrm{RelInt}(Q)\cap\overline{\psi_P(Z_+(f))}$ must be the set of limit points of some collection of non-compact components of $Z_+(f)$.

To see why a component of $\mathrm{RelInt}(Q)\cap\overline{\psi_P(Z_+(f))}$ can be the limit set of just one non-compact component of $\psi_P(Z_+(f))$, we can specialize the monomial change of coordinates from the proof of our last lemma as follows: Let w be any nonzero inner facet normal vector of Q and let A be any invertible $n\times n$ matrix such that Aw is the first standard basis vector. Also let δ be the minimum value of $w\cdot a$ as a ranges over $\mathrm{Supp}(f)$. Lemma 14 and Proposition 2 then tell us that $\psi_P(C)$ is diffeomorphic to some non-compact component of $Z_+(g)$ where $g(y_1,\ldots,y_n)=$

$f(y^A) = y_1^\delta g_\delta(y_2,\ldots,y_n) + \sum_\alpha y_1^\alpha g_\alpha(y_2,\ldots,y_n)$, the sum ranges over $\{w\cdot a > \delta \mid a\in A\}$, and $y_1^\delta g_\delta(y_2,\ldots,y_n) = \mathrm{Init}_w(f)(y^A)$. Dividing out by y_1^δ, we then obtain by Proposition 2, the implicit function theorem [Rud76, Thm. 9.28, Pg. 224], and a simple induction on $\dim Q$ that every connected component of $U\cap\overline{\psi_P(Z_+(f))}$ is a connected $(n-1)$-dimensional **quasifold** [Pra01, BP02], for some neighborhood U of Q in P. So assertion (1) is proved, with the additional strengthening that for all $w\in\mathbb{R}^n\setminus\{\mathbf{O}\}$, every component of $\mathrm{RelInt}(P^w)\cap\overline{\psi_P(Z_+(\mathrm{Init}_{w'}(f)))}$ is the limit set of some unique non-compact component of $Z_+(\mathrm{Init}_{w'}(f))$, where $P^{w'}$ is a face of dimension $1+\dim P^w$ and $w'\in\mathbb{R}^n$.

To prove (2), note that $\psi_P(C)$ must be a non-compact subset of P and a closed subset of $\mathrm{Int}(P)$. Since P is compact, $\overline{\psi_P(C)}$ must therefore be compact and contain a point in ∂P. So now let Q be the face of highest dimension d such that $\overline{\psi_P(C)}$ intersects $\mathrm{RelInt}(Q)$. By assertion (1) (and the definition of a quasifold [Pra01, Sec. 1]), $\partial P\cap\overline{\psi_P(C)}$ must be an $(n-2)$-dimensional quasifold with only finitely many connected components. Lemma 14 then tells us that $d<n-1 \Longrightarrow \dim(Q\cap\overline{\psi_P(C)})<d$, since $\mathrm{Init}_w(f)$ is not identically zero. So if $d<n-1$ we must then have that $\partial P\cap\overline{\psi_P(C)} = \bigcup_{Q' \text{ a face of } P} \mathrm{RelInt}(Q')\cap\overline{\psi_P(C)}$ has dimension strictly less than $n-2$, thus contradicting assertion (1). So $d=n-1$ and assertion (2) is proved. ■

Note that the smoothness hypothesis of Lemma 15 in fact implies that every non-compact connected component of $Z_+(f)$ contains an $(n-1)$-dimensional manifold. Note also that the smoothness hypothesis (at least for w that are inner facet normals) is necessary for assertion (1).

Example 6. *Consider $f(x,y):=(x+y-1)(y-x+1)$. Then $Z_+(f)$ consists of a exactly 2 disjoint rays and $\mathrm{Init}_{(0,1)}(f)=-(x-1)^2$ has a degenerate root at 1. In particular, $(1,0)$ is a limit point of both the rays of this $Z_+(f)$.* ⋄

Proof of Theorem 3: Let $Z:=Z_+(f)$. By Lemma 15, the number of non-compact connected components of Z is no more than $\sum_w N'_w$ where the sum ranges over all unit inner facet normals of $P=\mathrm{Newt}(f)$ and N'_w is the number of connected components of $\mathrm{RelInt}(P^w)\cap\overline{\psi_P(Z_+(f))}$. By Lemma 14, $\sum_w N'_w = \sum_w N_w$, so the first part of theorem 3 is proved.

The second assertion is then a trivial consequence of the first via the definition of $P(n,m)$.

To conclude, assertion (0) of Theorem 2 easily implies that for $n\leq 2$ our penultimate bound specializes to exactly the number of points of $\mathrm{Supp}(f)$ on the boundary of $\mathrm{Newt}(f)$, regardless of whether Z is smooth or not. To halve this bound, simply note that for smooth Z, every non-compact component C of Z is homeomorphic to an open interval. Therefore, by Lemma 15, $\overline{\psi_P(C)}$ must intersect the boundary of P exactly twice. So we are done. ■

Remark 7. *Bertrand Haas has pointed out that the very last assertion of Theorem 3 (concerning non-compact connected components of m-nomial curves in $\mathbb{R}^2_+$), in the case of* **integral** *exponents, follows easily from work of Isaac Newton published in 1744 [New44, Book I, Chap. 3]. The relevant result of Newton relates Puiseux series and diagrams involving the portion of* Newt(f) *visible from the origin, and can also be found in [Coo59, Chap. II, Paragr. 1, Pg. 213].* ⋄

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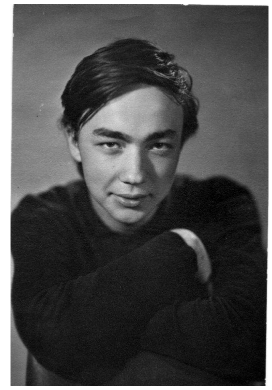
Konstantin Alexandrovich Sevast'yanov was born on January 21, 1956 in Astrakhan (an old Russian town on the Volga river) and graduated from a very famous mathematical high school organized by Andrey Nikolaevich Kolmogorov. At the age of 17 he was a winner of the International Mathematical Olympiad for high school students and thus skipped his entrance exams to become a mathematics student at Moscow State University. His supervisor was Anatoly Georgievich Kushnirenko, and Vladimir Igorevich Arnold and Askold Georgevich Khovanski also supervised Sevast'yanov's research. Sevast'yanov was a gifted student but suffered from poor health throughout his life. He formulated, around 1979, the key result that inspired Khovanski to create Fewnomial Theory. Sevast'yanov eventually went on permanent leave as his illness worsened and on December 7, 1984 he was killed after apparently being struck by a car.



Konstantin Alexandrovich Sevast'yanov, around 1983

Those who bless us with beautiful results should never be forgotten, even if tragedy obscures their accomplishments. We therefore dedicate this paper to the memory of Konstantin Alexandrovich Sevast'yanov.

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