4. Sets

4.1. The language of sets

- **Set Terminology and Notation**

  Set is a well-defined collection of objects.

  Elements are objects or members of the set.

Describing a Set

- **Roster notation:**

  $A = \{a, b, c, d, e\}$ Read: Set A with elements a, b, c, d, e.

- **Indicating a pattern:**

  $B = \{a, b, c, ..., z\}$ Read: Set B with elements being the letters of the alphabet.

If a is an element of a set A, we write $a \in A$ that read "a belongs to A." However, if a does not belong to A, we write $a \notin A$.

Set-builder notation (a more precise way of describing a set)

NOTATION 1. *Let $P(x)$ be a predicate. Then the notation*

$$\{x|P(x)\} \quad or \quad \{x : P(x)\}$$

denotes the set of all elements x such that $P(x)$ is a true statement. (The symbol "$|$" is read "such that".)

When D is a set,

$$\{x \in D|\ P(x)\} = \{x|\ x \in D \quad \wedge \quad P(x)\}$$

EXAMPLE 2. *Use set-builder notation and to describe the following sets in two different ways:*

a) $\mathbf{O}$

b) $5\mathbf{Z}$

c) $\mathbf{N}$

d) $\mathbf{Q}$

EXAMPLE 3. *Rewrite the following sets using roster notation:*

$A = \{x|x \in \mathbf{R} \wedge |x| = 1\} =$

$B = \left\{x|x \in \mathbf{R} \wedge x^4 = 1\right\} =$

$C = \left\{x|x \in \mathbf{C} \wedge x^4 = 1\right\} =$

Interval notation:

Intervals:

- bounded intervals:

1. closed interval $[a,b] =$
2. open interval $(a,b) =$
3. half-open,half-closed interval $(a,b] =$
4. half-closed,half-open interval $[a,b) =$

- unbounded intervals:

5. $[a,\infty) =$
6. $(a,\infty) =$
7. $(-\infty,a] =$
8. $(-\infty,a) =$
9. $(-\infty,\infty) =$

EXAMPLE 4. *Represent the following sets in interval notation when it is possible.*

a) $\{x \in \mathbf{R} |\ (x \geq 0) \wedge (x \in \mathbf{Z})\} =$

b) $\{x \in \mathbf{Z} |\ 3 \leq x < 10\} =$

c) $\{x \in \mathbf{R} |\ -2018 \leq x \leq 2019\} =$

Subsets

- Two sets, A and B, are **equal**, written $A = B$, if and only if they have exactly the same elements. (NOTE: they do not have to be in the same order!).
- If every element in set A is also an element in set B, then A is a subset of B, written $A \subseteq B$.
- If $A \subseteq B$, but $A \neq B$, then A is a **proper** subset of B, written $A \subset B$.

  Note that if $A = \{x \in D |\ P(x)\}$, then $A \subseteq D$.
- The **empty set** is the set that doesn't have any elements, denoted by $\emptyset$ or $\{\}$.
- The **universal set** is the set that contains all of the elements for a problem, denoted by U.

Using Symbols

Let $A, B \subseteq U$. Then

- $A = B \Leftrightarrow \forall x \in U, (x \in A \qquad x \in B)$
- $A \subseteq B \Leftrightarrow \forall x \in U, (x \in A \qquad x \in B)$
- $A \subset B \Leftrightarrow$
- $A \neq B \Leftrightarrow$

EXAMPLE 5. *Prove or disprove:* "If $B = \{x | x \in \mathbf{R} \wedge x^4 = 1\}$ and $C = \{x | x \in \mathbf{C} \wedge x^4 = 1\}$, then $B = C$."

EXAMPLE 6. *Let* $A = \{n \in \mathbf{Z} | n \textit{ is even}\}$, $B = \{n \in \mathbf{Z} | n^2 \textit{ is even}\}$, *and* $C = \{n^2 | n \textit{ is even}\}$. *Are these sets the same?*

EXAMPLE 7. *Let* $A = \{n \in \mathbb{Z} | n = 3t - 2 \textit{ for some } t \in \mathbb{Z}\}$ *and* $B = \{n \in \mathbb{Z} | n = 3t + 1 \textit{ for some } t \in \mathbb{Z}\}$. *Prove that* $A = B$.

Cardinality

infinite set

finite set

cardinality of A, $|A|$

EXAMPLE 8. *Let A and B be two sets.*

(a) ***TRUE/FALSE*** *If $A = B$, then $|A| = |B|$.*

(b) ***TRUE/FALSE*** *If $|A| = |B|$, then $A = B$.*

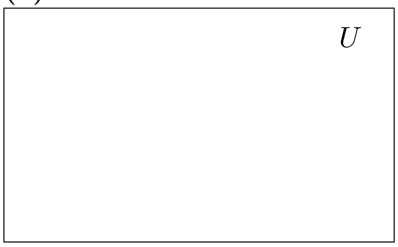
4.2 Operations on sets

VENN DIAGRAMS

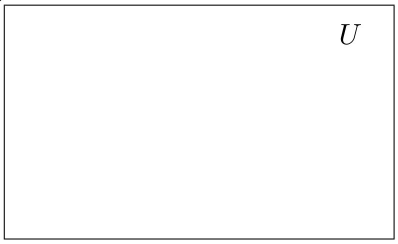
- a visual representation of sets (the universal set U is represented by a rectangle, and subsets of U are represented by regions lying inside the rectangle).

EXAMPLE 9. *Use Venn diagrams to illustrate the following statements:*

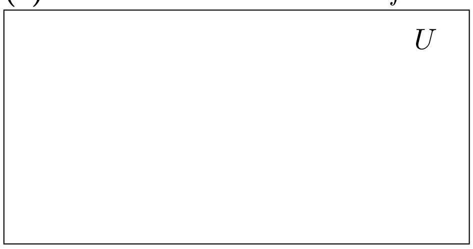
(a) $A = B$

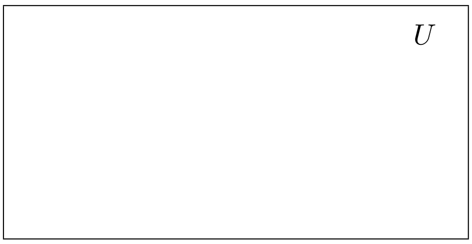


(b) $A \subset B \subset C$



(c) *A and B are not subsets of each other.*





DEFINITION 10. *Let A and B be sets in a universal set U. The* **union** *of A and B, written $A \cup B$, is the set of all elements that belong to either A or B or both. Symbolically:*

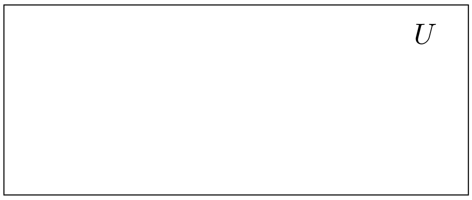
$$A \cup B = \{x \in U | x \in A \vee x \in B\}.$$

DEFINITION 11. *Let A and B be sets in a universal set U. The* ***intersection*** *of A and B, written $A \cap B$, is the set of all elements in common with A and B. Symbolically:*

$$A \cap B = \{x \in U | x \in A \ \wedge \ x \in B\}.$$

DEFINITION 12. *Let A and B be sets. The* **complement of** A **in** B *denoted $B - A$, is*

$$B - A = \{x \in U | x \in B \ \wedge \ x \notin A\}$$



REMARK 13. For convenience, if U is a universal set and A is a subset in U, we will write $U - A = \bar{A}$, called simply the **complement** of A.

EXAMPLE 14. *Let A be a subset of a universal set U. Prove the following*

(a) $\overline{\overline{A}} = A$.

(b) $\overline{\emptyset} = U$.

(c) $\overline{U} = \emptyset$

EXAMPLE 15. *Let $U = \{0, 1, 2, \ldots, 9, 10\}$ be a universal set, $A = \{0, 2, 4, 6, 8, 10\}$, and $B = \{1, 3, 5, 7, 9\}$. Find*

$$(\overline{A \cap B}) \cap (\overline{A \cup B}).$$

set notation	$=$	$\subset, \subseteq$	$\cup$	$\cap$	$\bar{\square}$	$\emptyset$
logical connectivity						

Power set

DEFINITION 16. *Let A be a set. The power set of A, written $P(A)$, is the following set*

$$P(A) = \{X \mid X \subseteq A\}.$$

EXAMPLE 17. *Find the following*

(a) $P(\{x, y\})$

(b) $|P(\{x, y\})|$

EXAMPLE 18. *Let $A = \{-1, 0, 1\}$.*

1. *Write all subsets of A.*

2. *Find all elements of power set of A.*

3. *Write 3 subsets of $P(A)$.*

4. *Find $|P(A)|$*

5. *Compute $|P(P(A))|$*

6. *What are $|P(A)|$ and $|P(P(A))|$ for an arbitrary set A?*

EXAMPLE 19. *Find*

(a) $P(\{\Delta\})$

(b) $P(\emptyset)$

(c) $P(P(\emptyset))$

(d) $P(\{\Delta, \square\})$

(e) $P(\{\emptyset, \{\emptyset\}\})$

REMARK 20. Note that

$$\emptyset \subseteq \{\emptyset, \{\emptyset\}\}, \quad \emptyset \subset \{\emptyset, \{\emptyset\}\}, \quad \{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}, \quad \{\emptyset\} \in \{\emptyset, \{\emptyset\}\},$$

as well as

$$\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset\}\}, \quad \{\{\emptyset\}\} \notin \{\emptyset, \{\emptyset\}\}, \quad \{\{\emptyset\}\} \in P(\{\emptyset, \{\emptyset\}\}).$$

Cartesian Product

DEFINITION 21. *Let A and B be sets. The* **Cartesian product** *of A and B, written $A \times B$, is the following set:*

$$A \times B = \{(a,b) |\ a \in A \wedge b \in B\}.$$

Informally, $A \times B$ is the set of **ordered** pairs of objects.

EXAMPLE 22. *Given $A = \{0,1\}$ and $B = \{4,5,6\}$.*

(a) *Does the pair $(6,1)$ belong to $A \times B$?*

(b) *List the elements of $A \times B$.*

(c) *What is the cardinality of $A \times B$?*

(d) *List the elements of $A \times A \times A$ and $(A \times A) \times A$.*

(e) *Does the triple $(1,6,4)$ belong to $A \times B \times B$?*

(f) *Describe the following sets $R \times R$, $R \times R \times R$.*

Fundamental properties of sets

THEOREM 23. *The following statements are true for all sets A, B, and C.*

1. $A \cup B = B \cup A$ (`commutative`)
2. $A \cap B = B \cap A$ (`commutative`)
3. $(A \cup B) \cup C = A \cup (B \cup C)$ (`associative`)
4. $(A \cap B) \cap C = A \cap (B \cap C)$ (`associative`)
5. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (`distributive`)
6. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (`distributive`)

   `DeMorgan's Laws:` *If A and B are the sets contained in some universal set U then*

7. $\overline{A \cup B} = \bar{A} \cap \bar{B}$.
8. $\overline{A \cap B} = \bar{A} \cup \bar{B}$.

Proving set properties

Use the following tautologies:

- $x \in A \cap B \Leftrightarrow (x \in A \wedge x \in B)$
- $x \in A \cup B \Leftrightarrow (x \in A \vee x \in B)$
- $x \in A - B \Leftrightarrow (x \in A \wedge x \notin B)$
- $A = B \Leftrightarrow (x \in A \Leftrightarrow x \in B)$
- $A \subseteq B \Leftrightarrow (x \in A \Rightarrow x \in B)$
- $x \in \bar{A} \Leftrightarrow x \notin A$
- $(x, y) \in A \times B \Leftrightarrow (x \in A \wedge y \in B)$

Methods:

- To prove $A \subseteq B$ it is sufficient to prove $x \in A \Rightarrow x \in B$.
- To prove $A = B$ it is sufficient to prove $x \in A \Leftrightarrow x \in B$.
- To prove $A = B$ it is sufficient to prove $A \subseteq B$ and $B \subseteq A$.
- To show that $A = \emptyset$ it is sufficient to show that $x \in A$ implies a false statement.

THEOREM 24. *The following statements are true for all sets A and B.*

1. $A \subseteq A \cup B$.
2. $A \cap B \subseteq A$.
3. *The empty set is a subset of every set. (Namely, for every set A, $\emptyset \subseteq A$. If $A \neq \emptyset$, then $\emptyset \subset A$.).*
4. $A \cup \emptyset = A$.
5. $A \cap \emptyset = \emptyset$.
6. $A \cap A = A \cup A = A$

COROLLARY 25.

EXAMPLE 26. *Let A and B be subsets of a universal set U. Show that $(A - B) \cap B = \emptyset$.*

PROPOSITION 27. *Let A and B be subsets of a universal set U. Then*

$$A - B = A \cap \bar{B}.$$

EXAMPLE 28. *Let A, B and C be sets. Prove that*

$$A - (B \cup C) = (A - B) \cap (A - C)$$

EXAMPLE 29. *For the sets A, B and C prove that*

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

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PROPOSITION 30. *Let* $A, B,$ *and* C *be sets, and suppose* $A \subseteq B$ *and* $B \subseteq C$*. Then* $A \subseteq C$*.*

EXAMPLE 31. *Let* A, B, C *and* D *be sets. If* $A \subseteq C$ *and* $B \subseteq D$*, then* $A \times B \subseteq C \times D$*.*

EXAMPLE 32. *Prove the following statement. Let* A *and* B *be subsets of a universal set* U*. Then* $A \subseteq B \Leftrightarrow A \cup B = B$*.*

EXAMPLE 33. *Let A and B be subsets of a universal set U. Prove that*

$$A = A - B \Leftrightarrow A \cap B = \emptyset.$$

4.3 Arbitrary unions and intersections

DEFINITION 34. *Let I be a set. An* ***indexed collection of sets*** $\{A_\alpha\}_{\alpha \in I}$ *represents a collection of sets such that for every $\alpha \in I$, there is a corresponding set A_α. In this case we call I the* **indexed set.**

Collection of sets	Indexed set	Shortened notation
$A_0, A_1, A_2, A_3, \ldots, A_{2016}$		
B_3, B_6, B_9, B_{77}		
$C_5, C_{10}, C_{15}, \ldots, C_{2015}$		

- **Union and Intersection**

EXAMPLE 35. *Complete the following*

(a) $x \in \bigcup_{\alpha \in I} A_\alpha \Leftrightarrow \exists \alpha \in I \ni x \in A_\alpha$

$x \notin \bigcup_{\alpha \in I} A_\alpha \Leftrightarrow$

(b) $x \in \bigcap_{\alpha \in I} A_\alpha \Leftrightarrow \forall \alpha \in I, x \in A_\alpha$

$x \notin \bigcap_{\alpha \in I} A_\alpha \Leftrightarrow$

EXAMPLE 36. *Given* $B_i = \{i, i+1\}$ *for* $i = 1, 2, \dots, 10$. *Determine the following*

(a) $\bigcap_{i=1}^{10} B_i$

(b) $B_i \cap B_{i+1}$

(c) $\bigcap_{i=k}^{k+1} B_i$ *where* $1 \leq k < 10$.

(d) $\bigcap_{i=j}^{k} B_i$ *where* $1 \leq j < k \leq 10$.

(e) $\bigcup_{i=j}^{k} B_i$ *where* $1 \leq j < k \leq 10$.

EXAMPLE 37. $A_n = \left\{x \in \mathbf{R} \mid -\frac{1}{n} \leq x \leq \frac{1}{n}\right\}, \quad n \in \mathbf{Z}^+$. *Find* $\bigcup_{n \in \mathbf{Z}^+} A_n$ *and* $\bigcap_{n \in \mathbf{Z}^+} A_n$.